FIXED POINTS AND OPENNESS IN A LOCALLY CONVEX SPACE

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Introduction. The problem of the openness of a mapping in a Banach space has been considered by a number of writers (cf. e.g. [1], [3]). It is an important problem which in many instances is connected with the problem of surjective mappings [4]. The main purpose of this paper is to generalize to locally convex spaces the following theorem.

THEOREM A (REICHBACH [3]). *Let F be a completely continuous mapping of a Banach space X into itself and let $A(x) = x - F(x)$. If for the point $y_0 \in A(X)$, there exists a point $x_0 \in A^{-1}(y_0)$ such that:*

$$(1) \quad \inf_{0 < r < \infty} M(x_0, r)/r < 1$$

where $M(x_0, r) = \sup_{\|x - x_0\| \leq r} \|F(x) - F(x_0)\|$ then $A(X)$ contains a ball with center y_0 and radius $r_0 \neq 0$.

It should be noted that our Theorem 3 below which generalizes Reichbach's Theorem A above to locally convex spaces does reduce, when the space considered is a normed space, to a stronger form of Theorem A since it is sufficient to take the $\sup \|F(x) - F(x_0)\|$ on the boundary of the ball rather than on the whole ball.

Our success in dealing with the present problem is due to an application of a recent generalisation by Singbal [6] of the Schauder-Tychonoff fixed-point theorem, which will be stated below in Theorem B.

1. Preliminaries. Throughout this paper, X will denote a Hausdorff, locally convex linear space, and by mapping we always understand a continuous mapping. Let P be the family of all seminorms defining the topology of X [2]. If $p \in P$, we call a p -ball with center x_0 and radius r and denote by $B_p(x_0, r)$ the closed and convex set $\{x \mid p(x - x_0) \leq r\}$; similarly we call the boundary of $B_p(x_0, r)$ which is precisely the closed set $S_p(x_0, r) = \{x \mid p(x - x_0) = r\}$ the p -sphere with center x_0 and radius r . We say that a self-map F of X is completely continuous [5] if F is continuous and if for a certain closed, convex, balanced neighborhood U of 0, the sets $[F(nU)]^-$ ($n = 1, 2, \dots$) are compact. If p is the gauge of U , i.e., $U = \{x \mid p(x) \leq 1\}$, we have $nU = \{x \mid p(x) \leq n\}$, and for definiteness F will be called p -completely

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continuous. For future use we state without proof the following simple lemmas:

LEMMA 1. *Let g be a p -completely continuous mapping of X into itself and let f be a self-mapping of X ; then the composite mapping $f \circ g: X \rightarrow X$ is p -completely continuous.*

LEMMA 2. *Let X_1 and X_2 be closed subsets of a topological space Z and let f_1 and f_2 be mappings of X_1 and X_2 into a topological space Y such that $f_1(x) = f_2(x)$ on $X_1 \cap X_2$. Then:*

$$\begin{aligned} f(x) &= f_1(x) && \text{for } x \in X_1, \\ &= f_2(x) && \text{for } x \in X_2 \end{aligned}$$

is a mapping of $X_1 \cup X_2$ into Y .

THEOREM B (SCHAUDER-TYCHONOFF-SINGBAL [6]). *A mapping of a closed, convex set of a Hausdorff and locally convex space X into a compact subset of itself has a fixed-point.*

2. The main theorem.

THEOREM 1. *Let f be a p -completely continuous mapping of X into X . If f maps the boundary of the p -ball $B_p(x_0, r)$ into the latter set, then f has a fixed-point in $B_p(x_0, r)$.*

PROOF. It is noted that the boundary of the p -ball $B_p(x_0, r)$ is precisely the p -sphere $S_p(x_0, r)$. Define the mapping $g: X \rightarrow X$ as follows:

$$\begin{aligned} g(x) &= f(x) && \text{if } p(x - x_0) \geq r, \\ g(x) &= f_1(x) = f(x) && \text{if } x \in X_1, \\ &= f_2(x) = x_0 + r \frac{f(x) - x_0}{p(f(x) - x_0)} && \text{if } x \in X_2, \end{aligned}$$

where

$$\begin{aligned} X_1 &= \{x \mid p(f(x) - x_0) \leq r\} \cap B_p(x_0, r), \\ X_2 &= \{x \mid p(f(x) - x_0) \geq r\} \cap B_p(x_0, r). \end{aligned}$$

Then by the continuity of f , the set X_1 and X_2 are closed subsets of $B_p(x_0, r)$; furthermore for $x \in X_1 \cap X_2$, $f_1(x) = f_2(x)$. Hence by Lemma 2, $g(x)$ is continuous on $B_p(x_0, r)$. Moreover on the p -sphere $S_p(x_0, r)$, we have $f_1(x) = f_2(x) = f(x)$, then another application of Lemma 2 shows that g is continuous on X .

Since $X \equiv \{x \mid p(x - x_0) \geq r\} \cup B_p(x_0, r)$, and f is p -completely con-

tinuous, it follows from Lemma 1 that g is also p -completely continuous. But $p(x - x_0) \leq r$ implies $p(x) \leq p(x_0) + r$; hence there exists a positive integer n such that $p(x) \leq p(x_0) + r \leq n$, i.e., $B_p(x_0, r) \subset nU$. It follows that $[g(B_p(x_0, r))] \subset [g(nU)]$ is compact. Hence, $g(x)$ maps the closed, convex set $B_p(x_0, r)$ into a compact subset of itself. By Theorem B, there exists a point $\bar{x} \in B_p(x_0, r)$ such that $g(\bar{x}) = \bar{x}$. Supposing $g(\bar{x}) = f_2(\bar{x})$, we get:

$$r \frac{f(\bar{x}) - x_0}{p(f(\bar{x}) - x_0)} = \bar{x} - x_0.$$

Hence, $p(\bar{x} - x_0) = r$, i.e., $\bar{x} \in S_p(x_0, r)$. Since $f(S_p(x_0, r)) \subset B_p(x_0, r)$, it follows that $p(f(\bar{x}) - x_0) \leq r$ and by the definition of $g(x)$ we have $\bar{x} = g(\bar{x}) = f(\bar{x})$.

COROLLARY. *Let A be a mapping of X into itself such that for some $\beta \neq 0$ the mapping $x - \beta A(x)$ is p -completely continuous. Suppose further that for some given $\bar{y} \in X$, the mapping $f(x) = x - \beta[A(x) - \bar{y}]$ maps the boundary of the p -ball $B_p(x_0, r)$ into the latter set, then there exists a point $\bar{x} \in B_p(x_0, r)$ such that $A(\bar{x}) = \bar{y}$.*

PROOF. Theorem 1 applies to the mapping $f(x) = x - \beta[A(x) - \bar{y}]$. In fact, since $x - \beta A(x)$ is p -completely continuous, it follows that $f(x)$ is also p -completely continuous. The condition $f(S_p(x_0, r)) \subset B_p(x_0, r)$ is satisfied, hence there exists $\bar{x} \in B_p(x_0, r)$ such that $f(\bar{x}) = \bar{x}$, $\beta \neq 0$, then $A(\bar{x}) = \bar{y}$. Q.E.D.

Let A be a mapping of X into a topological space Y and let $y_0 \in A(x)$, we say that $A(x)$ is open at the point y_0 if there exists a neighborhood V of y_0 in Y such that $A(X) \supset V$.

THEOREM 2. *Let A be a mapping of X into itself. Suppose that for the point $y_0 \in A(X)$ there exists an $r_0 > 0$ such that: For each $\bar{y} \in B_p(y_0, r_0)$, it is possible to find an $x_0 \in A^{-1}(\bar{y})$, a $\beta \neq 0$ and an $r > 0$ such that:*

- (i) $x - \beta A(x)$ is p -completely continuous,
- (ii) $f(S_p(x_0, r)) \subset B_p(x_0, r)$ where $f(x) = x - \beta[A(x) - \bar{y}]$.

Then A is open at the point $y_0 \in A(X)$.

PROOF. The proof is trivial. In fact, the corollary shows that for every $\bar{y} \in B_p(y_0, r_0)$, there exists $\bar{x} \in B_p(x_0, r)$ such that $A(\bar{x}) = \bar{y}$. Hence every $\bar{y}$ of $B_p(y_0, r_0)$ belongs to the image $A(X)$ of X , and therefore A is open at the point $y_0 \in A(X)$, and the theorem is proved.

As an illustration of Theorem 2, we prove the following important Theorem 3. First we pose the following definition: Let F be a mapping of X into itself, and let:

$$N_p(x_0, r) = \sup_{x \in S_p(x_0, r)} p(F(x) - F(x_0)).$$

THEOREM 3. *Let F be a p -completely continuous mapping of X into itself and let $A(x) = x - F(x)$. If there exists $x_0 \in X$ and a positive real number r such that:*

$$(3) \quad N_p(x_0, r) < r$$

then the mapping A is open at the point $y_0 = A(x_0)$.

PROOF. Since $F(x) = x - A(x)$ is p -completely continuous, condition 2 (i) of Theorem 2 is satisfying with $\beta = 1$. Let $f(x) = x - \beta[A(x) - \bar{y}]$, i.e., $f(x) = F(x) + \bar{y}$. Since $F(x)$ is p -completely continuous there exists a closed, convex and balanced neighborhood U of 0 in X satisfying $U = \{x \mid p(x) \leq 1\}$, such that the sets $[F(nU)]^-$ ($n = 1, 2, \dots$) are compact. We have:

$$\begin{aligned} p(f(x) - x_0) &= p(F(x) + \bar{y} - x_0) \\ (4) \quad &\leq p(F(x) - F(x_0)) + p(F(x_0) + \bar{y} - x_0) \\ &= p(F(x) - F(x_0)) + p(y_0 - \bar{y}) \end{aligned}$$

where $y_0 = A(x_0)$.

Let r be as in (3) and $r_1 < r$ such that for $x \in S_p(x_0, r)$ there is $p(F(x) - F(x_0)) \leq r_1 < r$. Choosing r_0 such that $0 < r_0 < r - r_1$, we obtain by (4) for every $\bar{y} \in B_p(y_0, r_0)$: $p(f(x) - x_0) \leq r_1 + (r - r_1)$ for every $x \in S_p(x_0, r)$, i.e., condition (2) (ii) of Theorem 2 is satisfied.

By Theorem 2, we have $A(X) \supset B_p(y_0, r_0)$ and hence A is open at $A(x_0)$.

REMARK 1. In a Banach space, with a norm $\| \cdot \|$, condition (3) becomes:

$$(3') \quad N(x_0, r) < r,$$

where

$$N(x_0, r) = \sup_{\|x - x_0\| = r} \|F(x) - F(x_0)\|.$$

This assumption is less restrictive than condition (1) in Reichbach's Theorem A, in fact, a completely continuous mapping on a ball $B(x_0, r) = \{x \mid \|x - x_0\| \leq r\}$ in a Banach space X into itself in general does not attain its maximum on the boundary of $B(x_0, r)$.

REMARK 2. If in Theorems 2 and 3, the mapping A is also linear, then the image $A(X)$ is an open linear subset of X , hence we have $A(X) = X$. This is an important case since we obtain for every

$\bar{y} \in X$ a solution of the equation $A(x) = \bar{y}$.¹

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¹ As remarked by the referee, for the linear case more elementary methods are available. The author is also indebted to the referee for many other helpful suggestions.