RETRACTION IN m-PARACOMPACT SPACES

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1. **Introduction.** The value of the adjunction space as an embedding space in the theory of absolute retracts for nonmetric spaces is well known [3]. Specifically, the adjunction space is crucial in proving that every AR(Q) is an ES(Q) for certain classes Q of normal spaces. However, since the topology imposed by identification on the adjunction space is sometimes difficult to work with, any method of proof which avoids getting involved directly with this topology would seem preferable.

In this paper we show that such a method is available for the class of m-paracompact normal spaces, and more generally for any classes of normal spaces which can be characterized by the normality of their product with a compact Hausdorff space. We are basically motivated by Morita's characterization of an m-paracompact normal space (Theorem 1). In Lemma 1 we establish a sort of distributive property over products which the attaching of spaces possesses, and we are then easily able to prove that the adjunction space of two m-paracompact normal spaces is m-paracompact normal. Then by some known techniques, it follows that if Q is the class of m-paracompact normal spaces and $X \in Q$, X is an AR(Q), resp. ANR(Q), if and only if X is an ES(Q), resp. NES(Q); and X is an AR(Q) if and only if X is a contractible ANR(Q).

2. **Preliminaries.** Let m be an infinite cardinal number. A space X is *m-paracompact* if every open cover of cardinality $\leqq m$ has a locally finite open refinement. We let I denote the closed unit interval $[0, 1]$, and I^m denotes the product space of m copies of I.

The reader is referred to [6] for some extensive results on m-paracompact spaces, and to [3] for definitions and basic properties of an AR(Q), resp. ANR(Q), i.e., absolute retract, resp. absolute neighborhood retract for a class Q of normal spaces and an ES(Q), resp. NES(Q), i.e., extension space, resp. neighborhood extension space for a class Q of normal spaces.

Morita [6, p. 229] characterizes an m-paracompact normal space in the following useful way:

THEOREM 1. *X is m-paracompact normal if and only if $X \times I^m$ is normal.*

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REMARK 1. For $m=\aleph_0$, Theorem 1 essentially reduces to a characterization of countable paracompactness by Dowker [**1**]: X is countably paracompact normal if and only if $X\times I$ is normal. For Hausdorff spaces, we get the result of Tamano [**7**]: X is paracompact Hausdorff if and only if $X\times Y$ is normal for every compact Hausdorff space Y.

At this point, we briefly recall the notion of an *adjunction space*: Let $f: C\to Y$ be a continuous map from a closed subset C of a space X into a space Y. If in the topological disjoint union $X\cup Y$, we identify $x\in C$ with $f(x)\in Y$, we obtain a quotient space $X\cup_f Y$ called the adjunction space of X and Y via the map f. In this construction, it is sometimes said that X is attached to Y by f and the map f is called the attaching map. We will denote an element of $X\cup_f Y$ by $[u]_f$.

For any space W, it is clear that the continuous map $f: C\to Y$ induces a continuous map $g: C\times W\to Y\times W$ by $g(x, w) = (f(x), w)$ for each $x\in C$ and $w\in W$. Now consider the topological disjoint union $(X\times W)\cup(Y\times W)$. Since $C\times W$ is closed in $X\times W$, we let g be the new attaching map which identifies $(x, w)\in C\times W$ with $g(x, w)\in Y\times W$. We then obtain the adjuction space of $X\times W$ and $Y\times W$ via g, namely, $(X\times W)\cup_g(Y\times W)$. An element of this new space will be denoted by $[u, w]_g$.

MAIN RESULTS. It is well known [**4**, p. 376] that the adjunction space preserves normality, i.e., if X and Y are normal then $X\cup_f Y$ is normal. In Lemma 2, we show that the adjunction space preserves m-paracompactness and normality. However, we first establish the following lemma.

LEMMA 1. *Let W be locally compact. Then $(X\cup_f Y)\times W$ is homeomorphic to $(X\times W)\cup_g(Y\times W)$.*

PROOF. The method of proof will be to construct continuous maps α: $(X\cup_f Y)\times W\to(X\times W)\cup_g(Y\times W)$ and β: $(X\times W)\cup_g(Y\times W)\to(X\cup_f Y)\times W$ such that $\beta\circ\alpha$ is the identity map on $(X\cup_f Y)\times W$ and $\alpha\circ\beta$ is the identity map on $(X\times W)\cup_g(Y\times W)$.

Let p_f: $X\cup Y\to X\cup_f Y$ and p_g: $(X\times W)\cup(Y\times W)\to(X\times W)\cup_g(Y\times W)$ be the natural projections of the disjoint unions $X\cup Y$ and $(X\times W)\cup(Y\times W)$ onto their respective quotient spaces. Since p_f is an identification and W is locally compact, the map $p_f\times 1$: $(X\cup Y)\times W\to(X\cup_f Y)\times W$ is also an identification [**2**, p. 262].

Define maps ϕ: $X\times W\to(X\cup_f Y)\times W$ by $\phi(x, w)=([x]_f, w)$ and ψ: $Y\times W\to(X\cup_f Y)\times W$ by $\psi(y, w)=([y]_f, w)$. It is easy to see that ϕ and ψ are continuous, and since $\phi(x, w)=([x]_f, w)=([f(x)]_f, w)$

$=\psi(g(x, w))$ for each $x\in C$ and $w\in W$, the maps ϕ and ψ are consistent.

Similarly, the restrictions $\sigma=p_g|X\times W$ and $\tau=p_g|Y\times W$ are continuous and since $\sigma(x, w)=[x, w]_g=[f(x), w]_g=\tau(f(x), w)$ for each $x\in C$ and $w\in W$, the maps σ and τ are consistent.

Now the common extension (ϕ, ψ) of the maps ϕ and ψ is just $p_f\times 1$. Likewise the common extension (σ, τ) of the maps σ and τ is just p_g. The consistency of the maps ϕ and ψ insures that $(p_f\times 1)\circ p_g^{-1}$ is single valued and so by the Transgression Theorem [2, p. 123] we have that the map $(p_f\times 1)\circ p_g^{-1}$ is continuous, and $p_f\times 1=(p_f\times 1)\circ p_g^{-1}\circ p_g$. Similarly, since $p_f\times 1$ is an identification, $p_g\circ(p_f\times 1)^{-1}$ is continuous, and $p_g=p_g\circ(p_f\times 1)^{-1}\circ(p_f\times 1)$. Now define

$$\alpha = p_g\circ(p_f\times 1)^{-1}, \qquad \beta=(p_f\times 1)\circ p_g^{-1}.$$

Then $\alpha\circ\beta$ and $\beta\circ\alpha$ are the required identity maps, and the proof is complete.

COROLLARY 1. *Let X and Y be m-paracompact normal spaces, C closed in X and f: $C\to Y$ continuous. Then $X\cup_f Y$ is m-paracompact normal.*

PROOF. By Theorem 1, both $X\times I^m$ and $Y\times I^m$ are normal. Since the adjunction space preserves normality, the space $(X\times I^m)\cup_g(Y\times I^m)$ is normal, where g is the map induced by f. By Lemma 1, $(X\cup_f Y)\times I^m$ is also normal and so by applying Theorem 1 again, $X\cup_f Y$ is m-paracompact normal.

REMARK 2. It should be clear that the technique we used in proving Corollary 1 would apply to any classes of normal spaces which can be characterized by the normality of their product with a compact Hausdorff space. In the Remark 1, we mentioned two such classes of normal spaces. Hence we easily get the following two corollaries, both of which are known results. The first is a result of Iseki [5, p. 443].

COROLLARY 2. *The adjunction space of two countably paracompact normal spaces is countably paracompact normal.*

The following corollary is a result of Hanner [3, p. 330].

COROLLARY 3. *The adjunction space of two paracompact (Hausdorff) spaces is paracompact (Hausdorff).*

Since X is paracompact if it is m-paracompact for every cardinal $m\geq\aleph_0$, another formulation of Corollary 3 is possible if we do not assume normality includes Hausdorff.

COROLLARY 4. *The adjunction space of two paracompact normal spaces is paracompact normal.*

Let Q be the class of m-paracompact normal spaces.

THEOREM 2. *Let* $X \in Q$. *Then* X *is an* AR(Q), *resp.* ANR(Q) *if and only if* X *is an* ES(Q), *resp.* NES(Q).

PROOF. This follows from Corollary 1, using the same method of proof of Theorem 8.1 in [**3**].

THEOREM 3. X *is an* AR(Q) *if and only if* X *is a contractible* ARN(Q).

PROOF. This follows from Theorem 2, using the same method of proof as in [**3**, p. 332]:

Let X be an AR(Q). By definition, X is m-paracompact normal and hence countably paracompact normal. Therefore, $X \times I$ is m-paracompact [**6**, p. 228] and normal [**1**, p. 222]. Consider the closed set $C = (X \times 0) \cup (X \times 1)$ of $X \times I$, and define a continuous map $f\colon C \to X$ by $f(x, 0) = x$, $f(x, 1) = x_0$, where x_0 is some point of X. By Theorem 2, X is an ES(Q) and so there exists an extension $F\colon X \times I \to X$ of f. The existence of F shows that X is contractible. Since an AR(Q) is an ANR(Q), the necessity is complete.

The sufficiency follows from Theorem 2 and a result of Hanner [**3**, p. 331].

REMARK 3. We conclude by thanking the referee for some helpful suggestions.

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