STATIONARY POINT FREE GROUP ACTIONS

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1. **Introduction.** The object of this paper is to establish the result:

THEOREM. *If $\alpha \in \Omega_*$ is a torsion element of the oriented cobordism ring Ω_* , then there is an oriented manifold M representing the cobordism class α together with an action ϕ of S^1 on M by orientation preserving diffeomorphisms having the property that $\phi(z, m) = m$, $z \in S^1$, $m \in M$, implies z is either 1 or -1 .*

Denote by $SF(G)$, G a compact Lie group, the ideal of classes $\alpha \in \Omega_*$ represented by an oriented manifold M on which G acts by orientation preserving diffeomorphisms with the action having no stationary points. One then has:

COROLLARY. *If T^k is the toroidal group $S^1 \times \cdots \times S^1$ (k -times), then $SF(T^k)$ is the torsion ideal of Ω_* .*

This result was conjectured by Conner and Floyd [2, p. 131] and was proved by P. G. Anderson (Thesis, Massachusetts Institute of Technology, Cambridge). Conner and Floyd proved in Corollary (43.8) that $SF(T^k) \subset \text{Torsion}(\Omega_*)$. Denoting by $\pi: T^k \rightarrow S^1$ the projection on the first factor and $\phi: S^1 \times M \rightarrow M$ the action given by the theorem,

$$T^k \times M \xrightarrow{\pi \times 1} S^1 \times M \xrightarrow{\phi} M$$

is a stationary point free action of T^k on M . Thus $\text{Torsion}(\Omega_*) \subset SF(T^k)$.

One also has the result conjectured by Conner and Floyd [2, p. 143]:

COROLLARY. *For $k \geq 2$, the ideal $SF(Z_2^k)$ is precisely the ideal of classes $\alpha \in \Omega_*$ having even Euler characteristic.*

The author is indebted to the National Science Foundation for financial support during this work and to Oxford University for their hospitality during part of this work.

2. **Proof of the theorem.** In order to prove the theorem, note first that if M is an oriented manifold and $\phi: S^1 \times M \rightarrow M$ is an S^1 action, then for each $t \in S^1$, $\phi(t, \cdot): M \rightarrow M$ is an orientation preserving diffeomorphism since it is homotopic to the identity. Thus the orientation preserving property is automatic for S^1 actions.

Received by the editors September 23, 1966.

Next, recall from Wall [4] that the algebra $\mathbb{W}_*$ of unoriented cobordism classes represented by a manifold with w_1 reduced integral is a polynomial algebra over Z_2 on classes x_{2k-1} , x_{2k} , x_{2j^2} with k not a power of 2. There is a homomorphism $\partial: \mathbb{W}_{n+1} \rightarrow \Omega_n$ obtained by sending the class of M into the class of the submanifold N of M dual to w_1 , and Wall has shown that image (∂) is precisely the set of torsion elements of Ω_* .

Using Wall's representative manifolds, one has that the classes x_{2k-1} and x_{2j^2} contain orientable manifolds, and since ∂ is a derivation, for any monomial

$$Y = \prod_{\alpha} x_{2^j \alpha} \cdot \prod_{\beta} x_{2k_{\beta}-1} \cdot \prod_{\gamma} x_{2k_{\gamma}}$$

that

$$\partial Y = \prod_{\alpha} x_{2^j \alpha} \cdot \prod_{\beta} x_{2k_{\beta}-1} \cdot \partial \left\{ \prod_{\gamma} x_{2k_{\gamma}} \right\}.$$

Thus, it suffices to show that each class $\partial \{x_{2k_1} \cdots x_{2k_r}\}$ contains a representative manifold on which S^1 acts in the desired fashion. There are two cases to be considered: $r=1$ and $r>1$.

First, consider the case $r=1$. Then $\partial x_{2k} = x_{2k-1}$ is representable by a Dold manifold $P(m, n)$ with m odd and n even (see [3]). In particular, $P(m, n)$ is obtained from $S^m \times CP(n)$ by identifying the points (x, η) and $(-x, \bar{\eta})$. Letting $m=2p-1$, define

$$\begin{aligned} S^1 \times S^m \times CP(n) &\rightarrow S^m \times CP(n): (z, (z_1, \dots, z_p), \eta) \\ &\rightarrow ((zz_1, \dots, zz_p), \eta) \end{aligned}$$

where $S^1 \subset C^1$, $S^m \subset C^p$ in the usual way. This preserves identifications to give

$$\phi: S^1 \times P(m, n) \rightarrow P(m, n)$$

and $\phi(z, u) = u$ implies z is either 1 or -1 . This gives an action of S^1 on $P(m, n)$ of the desired type.

For the case $r>1$, one wants a representative for $\partial(x_{2k_1} \cdots x_{2k_r})$. For this recall that a representative for x_{2k} is obtained as follows: $P(m, n)$ has an involution

$$\tau: P(m, n) \rightarrow P(m, n)$$

induced by the involution

$$\begin{aligned} S^m \times CP(n) &\rightarrow S^m \times CP(n): ((z_1, \dots, z_p), \eta) \\ &\rightarrow ((-z_1, \dots, -z_{p-1}, -\bar{z}_p), \eta) \end{aligned}$$

given by the reflection of S^m in the plane for which the last coordinate is zero. Then let $Q(m, n)$ be formed from $S^1 \times P(m, n)$ by identifying (t, u) and $(-t, \tau(u))$. As noted by Wall, $Q(m, n)$ represents x_{2k} (for properly chosen m and n).

Let $N_i = Q(m_i, n_i)$ represent x_{2k_i} , $i = 1, \dots, r$. Let $\pi_i: N_i \rightarrow S^1$ be the projection induced by $S^1 \times P(m_i, n_i) \rightarrow S^1: (t, u) \rightarrow t^2$. Then π_i is a fibration and realizes $w_1(N_i)$. Let S^1 act on N_i by means of the action

$$S^1 \times S^1 \times P(m_i, n_i) \rightarrow S^1 \times P(m_i, n_i): (z, t, u) \rightarrow (zt, u).$$

The action $\phi_i: S^1 \times N_i \rightarrow N_i$ has fixed points only for 1 and -1 in S^1 .

Now let $P: N_1 \times \dots \times N_r \rightarrow S^1: (v_1, \dots, v_r) \rightarrow (\pi_1(v_1) \dots \pi_r(v_r))$. This is the composition of the bundle maps

$$\pi_1 \times \dots \times \pi_r: N_1 \times \dots \times N_r \rightarrow S^1 \times \dots \times S^1$$

and

$$\mu: S^1 \times \dots \times S^1 \rightarrow S^1$$

where μ is the multiplication. Thus P is transverse regular on $1 \in S^1$. In addition, P realizes w_1 , so

$$V = \{(v_1, \dots, v_r) \in N_1 \times \dots \times N_r \mid \pi_1(v_1) \dots \pi_r(v_r) = 1\}$$

represents $\partial(x_{2k_1} \dots x_{2k_r})$. *Note.* This step was suggested by the proof of Anderson [1].

Let $\phi: S^1 \times V \rightarrow V: (z, (v_1, \dots, v_r)) \rightarrow (\phi_1(z^{-1}, v_1), v_2, \dots, v_{r-1}, \phi_r(z, v_r))$. Then ϕ is an action of S^1 and $\phi(z, v) = v$ implies z is 1 or -1 . Thus $\partial(x_{2k_1} \dots x_{2k_r})$ is represented by V on which S^1 acts in the desired manner.

3. Actions of Z_{2^k} .

PROPOSITION. *Let Z_{2^k} , $k \geq 1$, act on M without stationary points. Then the Euler characteristic, $\chi(M)$, is even.*

PROOF. One inducts on k . Let $F \subset M$ be the set of points fixed under the action of $Z_2 \subset Z_{2^k}$. By [2, (27.2)], $\chi(M) \equiv \chi(F) \pmod{2}$. Then the action of Z_{2^k} on F factors through an action of $Z_{2^{k-1}}$, so $Z_{2^{k-1}}$ acts on F without stationary points.

To prove the corollary, it then suffices to show that

$$SF(Z_4) \supset \{\alpha \in \Omega_* \mid \chi(\alpha) \equiv 0 \pmod{2}\}.$$

First note that Z_2 acts freely on $2M$ by interchanging components, so $SF(Z_4) \supset SF(Z_2) \supset 2\Omega_*$ and by the theorem $SF(Z_4) \supset \text{Torsion}(\Omega_*)$. Since $\Omega_*/(2\Omega_* + \text{Torsion}(\Omega_*))$ is isomorphic to the Z_2 polynomial algebra on the classes of the complex projective spaces $CP(2k)$, it suffices to

show that some manifold cobordant to $CP(2k) \pm CP(2)^k$ has a Z_4 action by orientation preserving diffeomorphisms with no stationary points.

For this, one may follow Conner and Floyd [2, p. 142]. One has a Z_4 action on $CP(2k)$ given by

$$T([z_0, z_1, z_2, \dots, z_{2k-1}, z_{2k}]) = [\bar{z}_0, -\bar{z}_2, \bar{z}_1, \dots, -\bar{z}_{2k}, \bar{z}_{2k-1}]$$

with $T^4=1$. T is orientation preserving since any diffeomorphism of $CP(2k)$ preserves orientation. Further, the only stationary point of T is $[1, 0, \dots, 0]$. Then $T \times \dots \times T$ acts on $CP(2)^k$ with one stationary point, and the action at this point is the same as the action at the stationary point of T on $CP(2k)$. Deleting neighborhoods of these points and gluing together along the resulting boundaries gives a manifold V^{4k} cobordant to $CP(2k) \pm CP(2)^k$ on which Z_4 acts as required.

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