ENTIRE FUNCTIONS OF BOUNDED INDEX

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I. **Introduction.** An entire function, $f(z)$, has a Taylor expansion about any point a in the complex plane of the form

$$f(z) = \sum_{i=0}^{\infty} a_n(z-a)^n.$$

Since this series is absolutely convergent everywhere in the plane, the terms $|a_n|$ must approach 0. Consequently, there exists for each a, an index $n_0 = n(a)$ for which $|a_n|$ is a maximal coefficient. B. Lepson [2] raised the problem of characterizing entire functions for which $n(a)$ is bounded. The latter are called functions of bounded index. In what follows, we shall give a partial solution to Lepson's problem. We shall also include a number of results using somewhat different conditions than those suggested by Lepson.

II. **Functions of bounded index and nonuniform bounded index.**

DEFINITION 1. An entire function is said to be of bounded index if and only if there exists an integer N, such that for all z

$$\max\left(|f|, |f^{(1)}|, \frac{|f^{(2)}|}{2!}, \cdots, \frac{|f^{(N)}|}{N!}\right) \geqq \frac{|f^{(j)}|}{j!}; \qquad j = 0, 1, 2, 3, \cdots \tag{1}$$

($f^{(0)}$ denotes f). We shall say that f is of index N, if N is the smallest integer for which (1) holds. An entire function which is not of bounded index is said to be of unbounded index.

A function of bounded index satisfies

$$\sum_{i=0}^{N} \frac{|f^{(i)}|}{i!} \geqq \frac{|f^{(j)}|}{j!}; \qquad j = 0, 1, 2, 3, \cdots. \tag{2}$$

Furthermore, if (2) holds then

$$\max\left(|f|, |f^{(1)}|, \frac{|f^{(2)}|}{2!}, \cdots, \frac{f^{(N)}}{N!}\right) \geqq \frac{1}{(N+1)} \frac{|f^{(j)}|}{j!}; \qquad j = 0, 1, 2, 3, \cdots.$$

These facts suggest

DEFINITION 2. An entire function $f(z)$ is said to be of nonuniform

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bounded index if and only if there exist integers N_j, such that

$$\sum_{i=0}^{N} \frac{|f^{(i)}(z)|}{i!} \geqq c\frac{|f^{(j)}(z)|}{j!} \quad \text{for } |z| > N_j; \qquad j = 0, 1, 2, 3, \cdots$$

c is any fixed constant.

For bounded regions Lepson [2] proved that: If $f(z) \not\equiv 0$ is an entire function and R is any bounded set, then there is an integer N, such that for any z in R and any nonnegative integer n

$$\frac{|f^{(n)}(z)|}{n!} \leqq \max\left(|f(z)|, |f^{(1)}(z)|, \frac{|f^{(2)}(z)|}{2!}, \cdots, \frac{|f^{(N)}(z)|}{N!}\right).$$

The function e^z is obviously of bounded index. More generally, we prove the simple

THEOREM 1. *If f is entire and satisfies $f^{(k+1)}=f$, then it is of bounded index.*

PROOF. Write $j=q(k+1)+r$, $r\leqq k$. Then

$$\left|\frac{f^{(j)}}{j!}\right| = \left|\frac{f^{q(k+1)+r}}{(q(k+1)+r)!}\right| = \left|\frac{f^{(r)}}{(q(k+1)+r)!}\right| < \left|\frac{f^{(r)}}{r!}\right|$$
$$\leqq \max\left(|f|, |f^{(1)}|, \cdots, \left|\frac{f^{(k)}}{k!}\right|\right)$$

and Theorem 1 follows.

As another generalization of the case $f(z)=e^z$ we prove:

THEOREM 2. *$f=Qe^{az}$ is of bounded index whenever Q is a polynomial.*

PROOF. We may assume without any loss of generality that the leading coefficient of Q is 1. Let

$$p_k = \frac{f^{(k)}}{e^{az}} = \sum_{i=0}^{n} \overset{k}{c_i} Q^{(i)} a^{k-i}; \qquad \overset{k}{c_i} = \frac{k(k-1)\cdots(k-i+1)}{i!} < k^n$$

for $1\leqq i\leqq k$; $c_0^k=1$ and $c_i^k=0$ for $i>k$, where n is the degree of Q.

$|Q^{(i)}| < 3r^n/2$ for all nonnegative integers i and for sufficiently large r; say $r>r_0$.

Assume that f is of unbounded index. Then there exist an infinite sequence of k and a corresponding sequence z_k both going to ∞ such that

$$|f(z_k)| < |f^k(z_k)|/k!.$$

This implies that

$$(3)\qquad |Q(z_k)| < \sum_{i=0}^{n} \frac{|\overset{k}{c}_i Q^{(i)}(z_k) a^{k-i}|}{k!}.$$

Hence, it suffices to show that (3) is impossible when $|a| > 1$, i.e., to show that we cannot have

$$(4)\qquad |Q(z_k)| < \sum_{i=0}^{n} \frac{|\overset{k}{c}_i| \, |Q^{(i)}(z_k)| \, |a|^k}{k!} < \frac{3}{2} \frac{k^n r_k^n |a|^k}{k!},$$

where $r_k = |z_k|$.

Since $|Q| > r^n/2$ for sufficiently large r, say $r > r_1$, (r_1 independent of k, of course) it follows that (4) would imply that

$$(5)\qquad 1 < 3k^n |a|^k / k!$$

for an infinite sequence of $k \to \infty$. Since (5) is impossible our theorem follows.

One might conjecture that $f(z)$ is of bounded index if and only if $f(z^k)$ is, where k is a positive integer. This, however, is false, as illustrated by the pair e^z and e^{z^2}; since e^z is of bounded index and e^{z^2} is not, as we shall see in the next section. Nevertheless one can prove the following:

THEOREM 3. *Let t be a positive integer. If $g(z) = f(z^{1/t})$ is entire and $f(z)$ is of bounded index (say of index k) then $g(z)$ is of nonuniform bounded index.*

PROOF. We may assume that $t > 1$. Using mathematical induction and performing some elementary calculations one can verify that for any positive integer n

$$(6)\qquad g^{(n)}(z) = p_{1n} f^{(1)}(z^{1/t}) + p_{2n} f^{(2)}(z^{1/t}) + \cdots + p_{nn} f^{(n)}(z^{1/t}),$$

where $p_{in} = c_{in}(z^{-1/t})^{nt-i}$, c_{in} a nonzero constant; $n = 1, 2, 3, \cdots$.

Using (6) one shows easily that

$$(7)\qquad \begin{aligned} &|g(z)| + |g'(z)| + \frac{|g^{(2)}(z)|}{2!} + \cdots + \frac{|g^{(k)}(z)|}{k!} \\ &\geqq |f(z^{1/t})| + \left(|\overset{*}{p}_{11}| - \sum_{j=2}^{k} |\overset{*}{p}_{1j}| \right) |f^{(1)}(z^{1/t})| \\ &+ \left(|\overset{*}{p}_{22}| - \sum_{j=3}^{k} |\overset{*}{p}_{2j}| \right) \frac{|f^{(2)}(z^{1/t})|}{2!} + \cdots \\ &+ \left(|\overset{*}{p}_{ii}| - \sum_{j=i+1}^{k} |\overset{*}{p}_{ij}| \right) \frac{|f^{(i)}(z^{1/t})|}{i!} + \cdots + |\overset{*}{p}_{kk}| \frac{|f^{(k)}(z^{1/t})|}{k!}. \end{aligned}$$

The p_{ij}^*'s are the same as the p_{ij}'s except for some positive constant multipliers.

On the other hand, for any integer $s>k$

$$\frac{|g^{(s)}(z)|}{s!} \leqq \sum_{i=1}^{s} |p_{is}^*| \frac{|f^{(i)}(z^{1/t})|}{i!}. \tag{8}$$

Thus to prove Theorem 3 it suffices to show that

$$\begin{aligned} &|f| + \left(|p_{11}^*| - \sum_{j=2}^{k} |p_{1j}^*| - |p_{1s}^*|\right)|f^{(1)}| \\ &\quad + \left(|p_{22}^*| - \sum_{j=3}^{k} |p_{2j}^*| - |p_{2s}^*|\right)\frac{|f^{(2)}|}{2!} + \cdots \\ &\quad + \left(|p_{ii}^*| - \sum_{j=i+1}^{k} |p_{ij}^*|\right)\frac{|f^{(i)}|}{i!} + \cdots \\ &\quad + \left(|p_{kk}^*| - |p_{ks}^*|\right)\frac{|f^{(k)}|}{k!} \\ &> |p_{(k+1)s}^*| \frac{|f^{(k+1)}|}{(k+1)!} + |p_{(k+2)s}^*| \frac{|f^{(k+2)}|}{(k+2)!} + \cdots + |p_{ss}^*| \frac{|f^{(s)}|}{s!} \end{aligned} \tag{9}$$

for sufficiently large $|z|=r$.

The terms $|p_{ii}^*|$ dominate the coefficients of $|f^{(i)}|$ (which we denote by c_i), since they have minimal degree (in the variable $z^{-1/t}$) in each term. Using this fact one can easily show that for any $c>0$

$$\frac{1}{s}\sum_{i=0}^{k}\left|\frac{c_i}{p_{(k+j)s}^*}\right|\frac{|f^{(i)}|}{i!} > c\sum_{i=1}^{k}\frac{|f^{(i)}|}{i!} > c\,\frac{|f^{(k+j)}|}{(k+j)!} \tag{10}$$

for $j=1, 2, \cdots, s-k$, and $|z|>N(s)$; $N(s)$ some integer dependent on s. The last inequality follows from the hypotheses of our theorem.

From (10) we get

$$s\left(\frac{1}{s}\sum_{i=0}^{k}|c_i|\frac{|f^{(i)}|}{i!}\right) > c\sum_{j=1}^{s-k}|p_{(k+j)s}^*|\frac{|f^{(k+j)}|}{(k+j)!}$$

for $|z|>N(s)$ and our proof is complete.

As an application of Theorem 3 we have

COROLLARY. *Let k be a positive integer. The entire function*

$$f(z) = \sum_{n=0}^{\infty}\frac{z^n}{(kn)!}$$

is of nonuniform bounded index.

PROOF. $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} \frac{\left(z^{1/k}\right)^{kn}}{(kn)!} = g(z^{1/k}),$$

where $g(z)$ satisfies $g^{(k)}(z) = g(z)$. Thus, by virtue of Theorem 1, $g(z)$ is of bounded index. Hence, by Theorem 3, $f(z)$ is of nonuniform bounded index.

It is very likely that Theorem 3 can be generalized to the following

CONJECTURE. If $g(z)$ is of unbounded index and $p(z)$ is a polynomial then $g(p(z))$ is also of unbounded index.

III. **Functions of unbounded index.** We begin with the following lemma due to E. Borel [1].

LEMMA 1. *If $V(r)$ is monotonic increasing, then for any $\epsilon' > 0$ and $\epsilon > 0$*

$$V\left(r + \frac{1}{(\log V(r))^{1+\epsilon'}}\right) \leqq (1+\epsilon) V(r)$$

for all r outside a set of finite measure.

PROOF. See Hayman [1, Lemma 2.4 (i), pp. 38–39], for the proof of a similar result.

LEMMA 2. *For any entire function $f(z)$ and any $\epsilon > 0$*

$$M_{f^{(n)}}(r) \leqq M_f(r)^{1+\epsilon}$$

for all r outside a set of finite measure. This exceptional set may depend on n.

PROOF. It is well known (see [3]) that for $R > r$

$$M_{f'}(r) \leqq \frac{M_f(R)}{R - r} \cdot \left(M_f(r) \text{ denotes } \max_{|z|=r} |f(z)| \right).$$

Choose

$$R = r + \frac{1}{(\log M_f(r))^{1+\epsilon'}}; \qquad \epsilon' > 0.$$

We get by virtue of Lemma 1 that

$$\begin{aligned} M_{f'}(r) &\leqq M_f\left(r + \frac{1}{(\log M_f(r))^{1+\epsilon'}}\right) \cdot (\log M_f(r))^{1+\epsilon'} \\ &< (1+\epsilon) M_f(r) (\log M_f(r))^{1+\epsilon'} \end{aligned} \tag{11}$$

for all r outside a set of finite measure.

Thus the assertion of our lemma is valid when $n=1$.

Suppose that $M_{f^{(k)}}(r) \leqq M_f(r)^{1+\epsilon}$. Then by (11)

$$\begin{aligned} M_{f^{(k+1)}}(r) &\leqq (1+\epsilon)M_{f^{(k)}}(r)(\log M_{f^{(k)}}(r))^{1+\epsilon'} \\ &\leqq (1+\epsilon)^3 M_f(r)^{1+\epsilon}(\log M_f(r))^{1+\epsilon'} < M_f(r)^{1+\epsilon''} \qquad (\epsilon'' > \epsilon) \end{aligned}$$

for all r outside a set of finite measure. Our lemma follows.

We now list some additional obvious properties of the maximum modulus function that we shall use in the sequel:

(a) $M_{fg}(r) \leqq M_f(r)M_g(r)$.

(b) $M_{f^n}(r) = M_f(r)^n$.

(c) If $M_g(r) < \epsilon M_f(r)$ for sufficiently large r, then for any $\epsilon > 0$, $(1-\epsilon)M_f(r) < M_{f+g}(r) < (1+\epsilon)M_f(r)$ for sufficiently large r.

(d) If $f(z)$ is transcendental, then for any $\epsilon > 0$, $M_f(r) < M_{f'}(r)^{1+\epsilon}$ for sufficiently large r.

THEOREM 4. *Let α and ϕ be any two transcendental entire functions and let δ be a real number less than $\frac{1}{2}$. If*

$$M_\alpha(r) < M_\phi(r)^\delta$$

for a set, s, of r of infinite measure, then $f = \alpha e^\phi$ is of unbounded index.

If α is a polynomial and ϕ is entire then f is of unbounded index.

PROOF. We prove the first part of the theorem. The second part follows from a similar argument. It follows by mathematical induction that

$$\begin{aligned} f^{(n)} &= [(\phi')^n\alpha + p_n(\phi', \phi^{(2)}, \cdots, \phi^{(n)}, \alpha, \alpha', \cdots, \alpha^{(n)})]e^\phi \\ &= h_n e^\phi, \end{aligned} \tag{12}$$

where p_n is a polynomial in $\phi', \phi^{(2)}, \cdots, \alpha, \alpha', \cdots, \alpha^{(n)}$ whose degree in the ϕ's is less than or equal to $n-1$ and where at most one of $\alpha, \alpha', \cdots, \alpha^{(n)}$ appears in any one term.

It follows from (12) and the properties of the maximum modulus function that for any $\epsilon > 0$

$$M_{p_n}(r) \leqq M_{\phi'}(r)^{(n-1)(1+\epsilon)+(1+\epsilon)\delta}$$

for all r in the set s, with the possible exception of a set of finite measure. Choosing ϵ sufficiently small, we deduce from (12) that

$$\begin{aligned} (M_{\phi'}(r))^{n-\delta} - (M_{\phi'}(r))^{(n-1)(1+\epsilon)+\delta(1+\epsilon)} - M_{h_n}(r) \\ < M_{\phi'}(r)^{n+\delta} + M_{\phi'}(r)^{(n-1)(1+\epsilon)+(1+\epsilon)\delta} \end{aligned}$$

or for sufficiently small ϵ we have

$$(13)\qquad (1-\epsilon)(M_{\phi'}(r))^{n-\delta} - M_{h_n}(r) < (1+\epsilon)(M_{\phi'}(r))^{n+\delta}$$

for a set of r of infinite measure.

Now assume that f is of index k; then

$$(14)\qquad |f| + |f'| + \frac{|f^{(2)}|}{2!} + \cdots + \frac{|f^{(k)}|}{k!} > \frac{|f^{(m)}|}{m!} \qquad (m > k).$$

Hence, from (12) we find that

$$|\alpha| + |h_1| + \frac{|h_2|}{2!} + \cdots + \frac{|h_k|}{k!} > \frac{|h_m|}{m!}$$

and consequently

$$(15)\qquad M_\alpha(r) + M_{h_1}(r) + \frac{M_{h_2}(r)}{2!} + \cdots + \frac{M_{h_k}(r)}{k!} > \frac{M_{h_m}(r)}{m!}\cdot$$

From (13) and (15) we get for sufficiently small $\epsilon_0 > 0$

$$(16)\qquad M_\alpha(r) + M_{h_1}(r) + \frac{M_{h_2}(r)}{2!} + \cdots + \frac{M_{h_k}(r)}{k!} < (M_{\phi'}(r))^{k+\delta+\epsilon_0}$$

for a set, s', of r of infinite measure. On the other hand

$$(17)\qquad M_{h_m}(r)/m! > (M_{\phi'}(r))^{m-\delta-\epsilon_0}$$

for a subset of s' of infinite measure.

Since (16) and (17) contradict (15) our theorem follows.

COROLLARY. *An entire function with at most a finite number of zeros is of bounded index if and only if it is of the form $p(z)e^{az}$, where $p(z)$ is polynomial and a is a complex constant.*

It is worth noting that the proof of Theorem 4 can be used to generalize the results of that theorem to functions of the form $\alpha e^\phi + \beta$ provided that suitable growth restrictions are imposed on β.

The methods of this paper are elementary. One can probably get somewhat stronger results by using the Wiman-Valiron theory.

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