ON PERIODICITY IN LIMIT SETS

JAMES W. ENGLAND

The purpose of this paper is to give a necessary and sufficient condition in order that the S limit set of a point consists of exactly one periodic orbit which is both S -Lyapunov stable and S -asymptotically stable.

The general reference for most terms used in this paper is [2]. In this paper we assume (X, T, π) is a transformation group. X will always be assumed to be a uniform space where the Hausdorff topology in X is induced by the uniformity. T will be assumed to be generative, that is, T is isomorphic to $R^m \times I^n \times C$ where R is the additive group of real numbers, I is the additive group of integers, m and n are nonnegative integers, and C is a compact abelian group [4]. S will denote a closed replete semigroup in T . The S -limit set of $y \in X$, S_y , is defined by $S_y = \bigcap_{t \in S} \text{Cl}(ytS)$ [1].

DEFINITION 1. The orbit of a point $y \in X$ is said to be uniformly S -Lyapunov stable with respect to $B \subset X$ provided that for each index α of X there is an index β of X such that if $w \in B \cap yt'\beta$ for some $t' \in T$ then $wt \in yt'\alpha$ for all $t \in S$.

LEMMA 1. Let $\text{Cl}(yS)$ be compact. In order that the orbit of each point of S_y be uniformly S -Lyapunov stable with respect to $yS \cup S_y$ it is necessary and sufficient that the orbit of y be uniformly S -Lyapunov stable with respect to S_y .

PROOF. Assume the orbit of y is uniformly S -Lyapunov stable with respect to S_y and let $x \in S_y$. We will show that the orbit of x is uniformly S -Lyapunov stable with respect to $yS \cup S_y$. Let α an index of X be given and let β be a symmetric index of X with the property that $\beta^2 \subset \alpha$. There exists a symmetric index δ of X with the property that if $z \in S_y \cap yt'\delta^2$ for some $t' \in T$ then $zt \in yt't\beta$ for all $t \in S$. Let $z \in yS \cup S_y$ such that $z \in xt'\delta$ for some $t' \in T$. Then there exists a $t'' \in T$ such that if $z \in yS$, $z = yt''$ or if $z \in S_y$ then $yt'' \in z\delta$. In either case $yt''t \in zt\beta$ for all $t \in S$. However, $yt'' \in xt'\delta^2$ which implies that $yt''t \in xt't\beta$ for all $t \in S$. Thus, $zt \in xt't\alpha$ for all $t \in S$.

Assume that the orbit of each point of S_y is uniformly S -Lyapunov stable with respect to $yS \cup S_y$. Let α , an index of X , be given and let β be a symmetric index of X such that $\beta^2 \subset \alpha$. For each $x \in S_y$ there exists a symmetric index δ_x of X such that if $z \in (yS \cup S_y) \cap xt'\delta_x^2$ for

Received by the editors February 21, 1966 and, in revised form, October 17, 1966.

some $t' \in T$ then $zt \in xt't\beta$ for all $t \in S$. The collection $\{x\delta_x\}_{x \in X}$ is an open covering of S_y . Since S_y is a closed subset of $\text{Cl}(yS)$ it is compact. Hence, there exists a finite number of points $\{x_1, x_2, \dots, x_n\}$ of S_y such that $S_y \subset \bigcup_{i=1}^n x_i\delta_{x_i}$. Let δ be a symmetric index of X with the property that $\delta \subset \bigcap_{i=1}^n \delta_{x_i}$. Let $z \in S_y \cap yt'\delta$ for some $t' \in T$. There exists an integer $1 \leq j \leq n$ such that $z \in x_j\delta_{x_j}$. This implies $zt \in x_jt\beta$ for all $t \in S$. Since $\delta \subset \delta_{x_j}$, $yt' \in z\delta \subset z\delta_{x_j}$ which implies $yt' \in x_j\delta_{x_j}^2$. Thus, $yt't \in x_jt\beta$ for all $t \in S$. This statement together with the previous statement implies that $zt \in yt't\alpha$ for all $t \in S$. Thus the orbit of y is uniformly S -Lyapunov stable with respect to S_y .

COROLLARY 1. *If $\text{Cl}(yS)$ is compact and the orbit of y is uniformly S -Lyapunov stable with respect to S_y then S_y is a minimal set which is the closure of a uniformly almost periodic orbit.*

PROOF. We first show that S_y is a minimal set. Let $x, z \in S_y$ and α , an index of X , be given. Let β be a symmetric index of X with the property that $\beta^2 \subset \alpha$. Lemma 1 implies that the orbit of x is uniformly S -Lyapunov stable with respect to $yS \cup S_y$. Hence, there exists an index γ of X such that if $w \in (yS \cup S_y) \cap xt'\gamma$ for some $t' \in T$ then $wt \in xt't\beta$ for all $t \in S$. There exists a $q \in S$ such that $yq \in x\gamma$. This implies $yqt \in xt\beta$ for all $t \in S$. Since $S_y \subset \text{Cl}(y(qS))$ and $z \in S_y$ it follows that for some $b \in S$ $yqb \in z\beta$. However $yqb \in xb\beta$ thus $xb \in z\alpha$. This implies that $\text{Cl}(xT) = S_y$. Hence S_y is a minimal set.

We now show that if $x \in S_y$ then the orbit of x is uniformly almost periodic. Let α , an index of X , be given and let β be a symmetric index of X with the property that $\beta^2 \subset \alpha$. Since the orbit of x is uniformly S -Lyapunov stable with respect to S_y there exists an index δ of X such that if $z \in S_y \cap xt'\delta^2$ for some $t' \in T$ then $zt \in xt't\beta$ for all $t \in S$. S_y is a compact minimal set which implies that x is almost periodic [2, 4.05]. There exists a syndetic set $A \subset T$ such that $xa \in x\delta$ for all $a \in A$. We now show that the syndetic set A will satisfy the requirement of the orbit of x being uniformly almost periodic. Let $t \in T$ and $a \in A$ be given. From the continuity of the mapping there exists an index σ of X such that if $z \in x\sigma$ then $za \in xa\delta$. Once again using the almost periodicity of x there exists a syndetic set $B \subset T$ such that $xb \in x\gamma$ for all $b \in B$ where γ is an index of X with the property that $\gamma \subset \sigma \cap \delta$. Since B is syndetic, tB^{-1} is also syndetic in T . Thus, using [2, 6.17] it follows that $tB^{-1} \cap S \neq \emptyset$. Let $b \in B$ so that $tb^{-1} \in S$. Using the definition of B and γ it follows that $xb \in x\gamma \subset x\sigma$. Thus $xba \in xa\delta$. Since $a \in A$ it follows that $xba \in x\delta^2$. This implies $xta = xba(tb^{-1}) \in xtb^{-1}\beta$ or

$$(1) \quad xta \in xtb^{-1}\beta.$$

Using the definition of γ it follows that $xb \in x\delta$. This implies that $xb(tb^{-1}) \in (xtb^{-1})\beta$ or

$$(2) \quad xt \in xtb^{-1}\beta.$$

Combining (1) and (2) with the fact that β is symmetric implies that

$$xta \in xtb^2 \subset xta.$$

This proves that the orbit of x is uniformly almost periodic.

In the case $T = R$, this corollary generalizes a theorem of Nemyckii (cf. [3, 9.06]).

DEFINITION 2 (BAUM [1]). Let S be a replete semigroup in T and $x, y \in X$ with $x \neq y$. Then x and y are S -asymptotic ($x \uparrow_s y$) provided that for each index γ of X there is a $q \in S$ such that $xqp \in yqp\gamma$ for all $p \in S$.

DEFINITION 3. The orbit of a point $y \in X$ is uniformly S -asymptotically stable with respect to $B \subset X$ provided there exists an index α of X such that if $z \in B \cap yt\alpha$ for some $t \in T$ then $yt \uparrow_s zt'$ for some $t' \in T$.

THEOREM 1. Let $\text{Cl}(yS)$ be compact. If the orbit of y is uniformly S -Lyapunov stable with respect to S_y and uniformly S -asymptotically stable with respect to S_y then S_y consists of exactly one periodic orbit which is uniformly S -Lyapunov stable and uniformly S -asymptotically stable with respect to yS .

PROOF. It follows from the above corollary that S_y is a minimal set and that T is uniformly almost periodic on S_y .

Let $x \in S_y$. We have only to show that $xT = S_y = \text{Cl}(xT)$. Since S_y is compact it follows from [2, 3.09] that x is periodic.

Assume there exists a $z \in S_y$ such that $z \notin xT$. From the uniform S -asymptotic stability of the orbit of y there exists a symmetric index α of X such that if $w \in S_y \cap yt\alpha$ then $wt' \uparrow_s yt$ for some $t' \in T$. Since $x \in S_y$ and $S_y \subset \text{Cl}(yS)$ there exists a $t_1 \in S$ such that $x \in yt_1\alpha$. $S_y \subset \text{Cl}(y(t_1S))$ implies there exists a $t_2 \in S$ such that $z \in yt_1t_2\alpha$. Hence $xt' \uparrow_s yt_1$ and $zt'' \uparrow_s yt_1t_2$ for some $t', t'' \in T$. This implies that $zt'' \uparrow_s xt_2t'$. Since $z \notin xT$, $zt'' \neq xt_2t'$. Thus there exists an index γ of X such that $zt'' \notin (xt_2t')\gamma$.

Let β be a symmetric index of X with the property that $\beta^3 \subset \gamma$. Since T is uniformly almost periodic on S_y there exists a syndetic set $A \subset T$ such that if $w \in S_y$ then

$$(1) \quad wa \in w\beta$$

for all $a \in A$. By Lemma 1 it follows that there exists an index δ of X such that if $w \in S_y$ and $r \in S_y \cap (wt')\delta$ for some $t' \in T$ then $rp \in wt'\beta\beta$

for all $p \in S$. Since $zt'' \uparrow {}_s xt_2 t'$ there exists a $q \in S$ such that $(zt''qp) \in (xt_2 t'qp)\delta$ for all $p \in S$. Let $zt'' = z'$, $xt' = x'$ and $p_1 \in S$. From the preceding statement we have

$$(2) \quad (z'qp_1t) \in (x't_2qp_1t)\beta$$

for all $t \in S$. S is a replete semigroup in T thus qp_1S is also a replete semigroup in S . From [2, 6.14] it follows that $A \cap qp_1S \neq \emptyset$. If $qp_1 = q'$ then $q'b \in A \cap q'S$ for some $b \in S$. Using (1) and (2) it follows that

$$\begin{aligned} x't_2q'b &\in x't_2\beta, \\ z'q'b &\in z'\beta, \\ z'q'b &\in (x't_2q'b)\beta. \end{aligned}$$

This implies that $z' \in (x't_2)\beta^3 \subset (x't_2)\gamma$ which is a contradiction. Thus $xT = S_y$. However, S_y is compact and $\text{Cl}(xT) \subset S_y$ thus $xT = \text{Cl}(xT)$ which proves that x is periodic.

Lemma 1 implies that the orbit of x is uniformly S -Lyapunov stable with respect to yS . In the same manner as in the proof of this lemma it follows that the orbit of x is uniformly S -asymptotically stable with respect to yS .

DEFINITION 4. A point $y \in X$ is S -Lyapunov stable with respect to $B \subset X$ provided that for each index α of X there exists an index β of X such that if $w \in B \cap y\beta$ then $wt \in y\alpha$ for all $t \in S$.

DEFINITION 5. A point $y \in X$ is S -asymptotically stable with respect to $B \subset X$ provided there exists an index α of X such that if $z \in B \cap y\alpha$ then $y \uparrow {}_s zt'$ for some $t' \in T$.

THEOREM 2. Let $\text{Cl}(yS)$ be compact. If S_y consists of exactly one periodic orbit which is S -asymptotically stable and S -Lyapunov stable with respect to yS then the orbit of y is uniformly S -Lyapunov stable and uniformly S -asymptotically stable with respect to S_y .

PROOF. Let $x \in S_y$. Then $xT = xK$ for some compact subset K of T . Thus x being S -Lyapunov stable with respect to yS implies that its orbit is uniformly S -Lyapunov stable with respect to yS . By Lemma 1 the orbit of y is uniformly S -Lyapunov stable with respect to S_y . In the same manner it can be shown that the orbit of y is uniformly S -asymptotically stable with respect to S_y .

In case $T = R$ it can be shown that if $x \in S_y$ and $xT = S_y$ and x is S -asymptotically stable with respect to yS then the orbit of x is uniformly S -Lyapunov stable with respect to yS . The converse of this statement is also valid.

BIBLIOGRAPHY

1. J. D. Baum, *Asymptoticity in topological dynamics*, Trans. Amer. Math. Soc. **77** (1954), 506–519.
2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., Vol. 36, Amer. Math. Soc., Providence, R. I., 1955.
3. V. V. Nemyckii and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Math. Series, No. 22, Princeton Univ. Press, Princeton, N. J., 1960.
4. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Indust., No. 869, Hermann, Paris, 1940.

UNIVERSITY OF VIRGINIA