NONHEREDITARY SEMISIMPLE CLASSES

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1. Introduction. For a class $\mathfrak{M}$ of rings let $\mathfrak{g}_1\mathfrak{M} = \{K \mid K \text{ is an ideal of some } R \in \mathfrak{M}\}$ be called the *ideal class* of $\mathfrak{M}$, and $\mathfrak{h}\mathfrak{M} = \{K \mid K \text{ is a homomorphic image of some } R \in \mathfrak{M}\}$ the *homomorphic closure* of $\mathfrak{M}$. A class $\mathfrak{M}$ is *hereditary* if $\mathfrak{g}_1\mathfrak{M} = \mathfrak{M}$ and *homomorphically closed* if $\mathfrak{h}\mathfrak{M} = \mathfrak{M}$. A class which is both hereditary and homomorphically closed will be called a *universal class*. We will choose an arbitrary, but fixed, universal class $\mathcal{C}$ of (not necessarily associative) rings and assume that all rings considered are members of $\mathcal{C}$.

Note that, for $\mathfrak{M}$ an arbitrary class, $\mathfrak{h}\mathfrak{M}$ is homomorphically closed and is minimal in the sense that if $\mathfrak{M} \subseteq \mathfrak{N}$ where $\mathfrak{N}$ is homomorphically closed, then $\mathfrak{h}\mathfrak{M} \subseteq \mathfrak{N}$.

To construct a minimal hereditary class containing $\mathfrak{M}$, assume for induction that for a given integer $n \geq 2$, classes $\mathfrak{g}_j\mathfrak{M}$ have been defined for all $j < n$. Then define $\mathfrak{g}_n\mathfrak{M} = \mathfrak{g}_1\mathfrak{g}_{n-1}\mathfrak{M}$. By induction $\mathfrak{g}_n\mathfrak{M}$ exists for all n . Call $\mathfrak{g}\mathfrak{M} = \bigcup \mathfrak{g}_n\mathfrak{M}$ the *hereditary closure* of $\mathfrak{M}$. It is clear by its definition that $\mathfrak{g}\mathfrak{M}$ is hereditary and, by induction, any hereditary class containing $\mathfrak{M}$ also contains $\mathfrak{g}\mathfrak{M}$.

It is well known that if $\mathcal{O}$ is a radical class [1], [2] then

$$(1) \quad \mathfrak{s}\mathcal{O} = \{R \in \mathcal{C} \mid \mathfrak{g}_1R \cap \mathcal{O} = 0\}$$

is a semisimple class [3, p. 3]. (Note: we write R for the class $\{R\}$ containing R as its sole member.) On the other hand, if $\mathcal{Q}$ is a semisimple class [3, Theorem 2, p. 5] then

$$(2) \quad \mathfrak{u}\mathcal{Q} = \{R \in \mathcal{C} \mid \mathfrak{h}R \cap \mathcal{Q} = 0\}$$

is a radical class [3, p. 6], called the *upper radical class* defined by $\mathcal{Q}$. Note that the proofs of [3, Chapter I] apply equally well to non-associative rings.

We will also use the Kurosh construction (modified by Anderson, Divinsky and Sulinsky [3, Footnote, p. 12]) of the *lower radical* defined for a class $\mathfrak{M}$: Let $\mathfrak{M}_1 = \mathfrak{h}\mathfrak{M}$ and assume that for a given ordinal β , classes $\mathfrak{M}_\alpha$ have been defined for all ordinals $\alpha < \beta$. Then define $\mathfrak{M}_\beta = \{R \in \mathcal{C} \mid \text{every nonzero } K \in \mathfrak{h}R \text{ satisfies the condition } \mathfrak{g}_1K \cap \mathfrak{M}_\alpha \neq 0 \text{ for some } \alpha < \beta\}$. Then the lower radical $\mathfrak{L}\mathfrak{M} = \bigcup \mathfrak{M}_\beta$.

2. Condition that a semisimple class be nonhereditary. It is well known that if $\mathcal{C}$ is an associative class or an alternative class then

Received by the editors September 1, 1966.

every semisimple class is hereditary [4, Corollary 2, p. 597 and Corollary 2, p. 602]. The question has been raised as to whether or not there exists a nonhereditary semisimple class. The following theorem gives a necessary and sufficient condition for the existence of such a class:

THEOREM 1. *A universal class $\mathcal{C}$ contains a nonhereditary semisimple class if and only if there exists a ring $R \in \mathcal{C}$ such that $\mathfrak{g}_1 R \cap \mathfrak{L}J = 0$ for some nonzero ring $J \in \mathfrak{g}R$.*

PROOF. From the definition (1) it follows that $\mathfrak{g}_1 R \cap \mathfrak{L}J = 0$ implies $R \in \mathfrak{S}\mathfrak{L}J$. But for any radical $\mathfrak{L}J \cap \mathfrak{S}\mathfrak{L}J = 0$, so since $J \in \mathfrak{L}J$ it follows that $J \notin \mathfrak{S}\mathfrak{L}J$. However $J \in \mathfrak{g}R$ whence $\mathfrak{S}\mathfrak{L}J$ is not hereditary.

Conversely, suppose the semisimple class $\mathfrak{M}$ is nonhereditary. Then there exists a ring $R \in \mathfrak{M}$ such that $K \notin \mathfrak{M}$ for some $K \in \mathfrak{g}_1 R$. By [3, Theorem 2, p. 5] there exists some $J \in \mathfrak{g}_1 K$ such that $J \in \mathfrak{u}\mathfrak{M}$. By the minimality of the lower radical [3, Lemma 5, p. 13] this implies $\mathfrak{L}J \subseteq \mathfrak{u}\mathfrak{M}$. By the definition (2) of the upper radical, $\mathfrak{u}\mathfrak{M} \cap \mathfrak{g}_1 \mathfrak{M} = 0$. But $R \in \mathfrak{M}$ whence $\mathfrak{g}_1 R \subseteq \mathfrak{g}_1 \mathfrak{M}$ and so $\mathfrak{g}_1 R \cap \mathfrak{L}J = 0$.

3. Construction of nonhereditary semisimple classes. In this section rings will be constructed satisfying Theorem 1, with J having contrasting properties; namely in one case $J^2 = J$, and in the other case $J^2 = 0$. The class $\mathcal{C}$ in which these are to be constructed is the class of all (not necessarily associative) rings.

We will need to use specializations of the following lemma due to A. E. Hoffman [5]. For completeness the proof is supplied.

LEMMA 1. *Let $\mathfrak{N}$ be a hereditary class. If a class $\mathfrak{M}$ satisfies the condition $\mathfrak{K}\mathfrak{M} \cap \mathfrak{N} = 0$ then $\mathfrak{L}\mathfrak{M} \cap \mathfrak{N} = 0$.*

PROOF. $\mathfrak{M}_1 = \mathfrak{K}\mathfrak{M}$ satisfies $\mathfrak{M}_1 \cap \mathfrak{N} = 0$ and assume for induction that for a given ordinal β , $\mathfrak{M}_\alpha \cap \mathfrak{N} = 0$ for all ordinals $\alpha < \beta$. If there exists a nonzero $R \in \mathfrak{M}_\beta \cap \mathfrak{N}$, then by definition $\mathfrak{g}_1 R \cap \mathfrak{M}_\alpha \neq 0$ for some $\alpha < \beta$. But $\mathfrak{g}_1 \mathfrak{N} = \mathfrak{N}$ and since $R \in \mathfrak{N}$ it follows that $\mathfrak{g}_1 R \subseteq \mathfrak{N}$, contradicting the induction hypothesis. By transfinite induction $\mathfrak{M}_\beta \cap \mathfrak{N} = 0$ for all ordinals β and hence $\mathfrak{L}\mathfrak{M} \cap \mathfrak{N} = 0$.

A ring K will be called *hereditarily idempotent* if $H^2 = H$ for all $H \in \mathfrak{g}K$. A ring R with the property $R^2 = 0$ will be called a *zero ring*. It is clear that we can specialize $\mathfrak{N}$ in Lemma 1 to be either: (a) the class $\mathcal{D}$ of all hereditarily idempotent rings, or (b) the class $\mathcal{Z}$ of all zero rings.

Consider the ring K generated by nonassociative symbols $\{e, x, y\}$ over the field Z_2 of integers modulo 2, subject to the relations:

$e^2 = ex = xe = e$, $ey = ye = xy = yx = x$, $x^2 = y^2 = 0$. It is clear that the only proper ideal of K is $I = K^2 = \{0, e, x, e+x\}$, and that I contains the ideal $J = \{0, e\}$. Since $J^2 = J$ and J is simple, we have $\mathcal{Z} \cap \mathcal{K}J = 0$. Thus by Lemma 1, $\mathcal{Z} \cap \mathcal{L}J = 0$. But if any ring $R \neq R^2$ then $R \notin \mathcal{L}J$ for otherwise the radical class $\mathcal{L}J$ would contain the zero ring R/R^2 . Since the only nonzero ideals of K are K , for which $I = K^2 \neq K$, and I for which $J = I^2 \neq I$, it follows that $\mathcal{J}_1 K \cap \mathcal{L}J = 0$. Thus by Theorem 1, $\mathcal{S}\mathcal{L}J$ is a nonhereditary class.

Using this example, we can prove:

THEOREM 2. *The class $\mathfrak{J}$ of all idempotent rings is a radical class whose semisimple class $\mathcal{S}\mathfrak{J}$ is nonhereditary.*

PROOF. If $K \in \mathfrak{J}$ then $K^2 = K$ so that $(K\phi)^2 = K\phi$ for all homomorphisms ϕ . Thus $\mathfrak{J}$ is homomorphically closed. On the other hand, $K \notin \mathfrak{J}$ implies $K^2 \neq K$ so that $K/K^2 \in \mathcal{Z}$. Thus $\mathcal{J}_1(K/K^2) \subseteq \mathcal{J}_1\mathcal{Z} = \mathcal{Z}$. Since $\mathcal{Z} \cap \mathfrak{J} = 0$, it follows that $\mathcal{J}_1(K/K^2) \cap \mathfrak{J} = 0$, that is $K/K^2 \in \mathcal{S}\mathfrak{J}$. By [3, Theorem 1] $\mathfrak{J}$ is a radical class. Now the ring K constructed above has no idempotent ideals, so $K \in \mathcal{S}\mathfrak{J}$. But $J \in \mathcal{J}K$ with $J \in \mathfrak{J}$ so, by Theorem 1, $\mathcal{S}\mathfrak{J}$ is nonhereditary.

Now consider the ring R generated over $\mathbb{Z}_2$ by the symbols $\{u, v, w\}$ satisfying relations: $u^2 = 0$, $uv = vu = u$, $uw = wu = vw = wv = v^2 = v$, $w^2 = w$. It is clear that $I = \{0, u, v, u+v\}$ is the only proper ideal of R and that $J = \{0, u\}$ is an ideal of I . Since $J^2 = 0$ and J is simple, we have $\mathcal{D} \cap \mathcal{K}J = 0$. Thus by Lemma 1, $\mathcal{D} \cap \mathcal{L}J = 0$. But R/I , $I/J \in \mathcal{D}$ so that $\mathcal{J}_1 R \cap \mathcal{L}J = 0$. It follows from Theorem 1 that $\mathcal{S}\mathcal{L}J$ is nonhereditary.

From this construction follows:

THEOREM 3. *If $\mathfrak{B}$ is the lower Baer radical class (that is, $\mathfrak{B} = \mathcal{L}\mathcal{Z}$ where $\mathcal{Z}$ is the class of all zero rings), then $\mathcal{S}\mathfrak{B}$ is a nonhereditary semisimple class.*

PROOF. In the ring R constructed above $J \in \mathcal{Z}$ so that $\mathcal{L}J \subseteq \mathcal{L}\mathcal{Z} = \mathfrak{B}$. Now $\mathcal{Z}$ is homomorphically closed and $\mathcal{D} \cap \mathcal{Z} = 0$ so by Lemma 1, $\mathcal{D} \cap \mathcal{L}\mathcal{Z} = 0$. Since every ideal of R has an image in $\mathcal{D}$, it follows that $R \in \mathcal{S}\mathfrak{B}$. But $J \in \mathfrak{B}$ so $\mathcal{S}\mathfrak{B}$ is nonhereditary.

Another consequence of this construction is: *The class $\mathcal{P}$ of all prime rings is nonhereditary.* This follows from the fact that the ring R just constructed is prime, whereas J is not.

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ON COHOMOLOGICAL TRIVIALITY

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Let G be a finite group and A be a G -module. It is well known that if for some two consecutive dimensions the cohomology groups $H^r(H, A)$ are trivial for all subgroups H of G , then $H^r(H, A)$ are trivial for all dimensions r and for all subgroups H of G . This note is to point out the following generalization.

THEOREM. *Let G be a finite group and A be a G -module. If for some integer k and for some odd positive integer d , $H^k(H, A)$ and $H^{k+d}(H, A)$ are trivial for all subgroups H of G , then $H^r(H, A)$ are trivial for all integers r and for all subgroups H of G .*

PROOF. As usual we proceed by induction on the order $n = |G|$. The theorem is trivial for $n = 1$. Suppose $n > 1$ and assume the truth of the theorem for all groups of order $< n$. In particular, we may assume that $H^r(H, A) = 0$ for every dimension r and for every proper subgroup H of G . If n is not a prime power, then every Sylow subgroup of G is a proper subgroup and the conclusion follows by a well-known result.

Suppose now n is a prime power, so that G is solvable. Let H be a proper normal subgroup of G such that the quotient group G/H is cyclic. Since $H^r(H, A) = 0$ for every r , we have the Fundamental Exact Sequence of Hochschild and Serre [2];

$$0 \rightarrow H^r(G/H, A^H) \rightarrow H^r(G, A) \rightarrow H^r(H, A)$$

for every $r > 0$. Since the last term is trivial, this says that $H^r(G/H, A^H) \cong H^r(G, A)$ for every $r > 0$. By dimension shifting we could have assumed that $k > 0$. Thus we have that $H^k(G/H, A^H) = H^{k+d}(G/H, A^H) = 0$. Since G/H is cyclic and d is odd, we obtain