GROUPS OF ELLIPTIC LINEAR FRACTIONAL TRANSFORMATIONS

R. C. LYNDON AND J. L. ULLMAN

1. Introduction. Our aim is to give a simple and self contained statement and proof of the following theorem:

Every subgroup of the group of linear fractional transformations of the complex plane, which contains only elliptic transformations, is conjugate to the image under stereographic projection of a group of rotations of a sphere.

An important consequence is that every discontinuous (or discrete) elliptic group is finite.

Simple proofs of these well-known results do not seem to be easily accessible. Indeed, we are indebted to J. Lehner for the only reference we know, to Fatou [1], where an argument similar to that given here is presented in a much fuller context.

2. Definitions and preliminaries. A *linear fractional transformation* is a map $T: z \rightarrow (az+b)/(cz+d)$, of the (extended) complex plane onto itself, with complex coefficients and nonvanishing determinant. We may require $ad-bc=1$; then the matrix is determined up to a factor -1 , and we may write

$$T = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These transformations form a group $\mathfrak{L}$, with composition corresponding to matrix multiplication.

An *elliptic transformation* is one that is conjugate to a rotation $z \rightarrow kz$, $|k|=1$, of the complex plane. More explicitly, T is elliptic if $T=I$, the identity map $z \rightarrow z$, or if it has distinct eigenvalues λ and $\lambda^{-1}=\bar{\lambda}$ (complex conjugate) with $|\lambda|=1$. Then T has trace $a+d=\lambda+\bar{\lambda}$ real, with $|a+d| \leq 2$. It follows also that T has two distinct fixed points.

Consider the finite complex plane as a plane in three dimensional space with coordinate axes (x, y, t) forming a right handed system. Let Σ be the unit sphere, with equator the circle $|z|=1$ and north pole N . *Stereographic projection* carries each point $P \neq N$ on Σ into the point P' where the line NP meets the plane, with $N' = \infty$. If $\tilde{T}$ is a rotation of Σ ; there is a unique transformation T in $\mathfrak{L}$ such that, when $\tilde{T}(P) = Q$, $T(P') = Q'$. Let $\mathfrak{R}$ be the group of all linear fractional

Received by the editors August 18, 1966.

transformations T obtained in this way from a rotation $\tilde{T}$ of Σ .

Let $\tilde{A}_\theta$ denote the rotation of Σ around the t axis which turns the positive x axis by an angle θ towards the positive y axis. The corresponding element A_θ in $\mathfrak{R}$ is

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

Let $\tilde{B}_\phi$ denote the rotation of Σ around the y axis which turns the positive x axis by an angle ϕ towards the positive t axis. The corresponding element B_ϕ in $\mathfrak{R}$ is

$$\begin{pmatrix} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} \\ -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}.$$

A general rotation $\tilde{T}$ of Σ will have a fixed point P whose spherical coordinates are (ϕ, θ) , and a rotation angle α . If O is the origin, ϕ is the angle between ON and OP satisfying $0 \leq \phi \leq \pi$, while θ is the angle between the positive x axis and the projection of OP in the xy plane measured counterclockwise and satisfying $0 \leq \theta < 2\pi$. We take $0 \leq \alpha < 2\pi$ as the clockwise angle of rotation as viewed from the center of the sphere. Now T can be written as the composition $(\tilde{C})^{-1} \tilde{A}_\alpha \tilde{C}$ where $\tilde{C} = \tilde{B}_{-\phi} \tilde{A}_{-\theta}$ takes P to N . The composition of the corresponding elements of L yields

$$\begin{pmatrix} \cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \cos \phi & i \sin \frac{\alpha}{2} e^{i\theta} \sin \phi \\ i \sin \frac{\alpha}{2} e^{-i\theta} \sin \phi & \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \cos \phi \end{pmatrix}.$$

We note that T in $\mathfrak{R}$ has the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with $|a|^2 + |b|^2 = 1$. Conversely, if an element T of $\mathfrak{L}$ is of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with $|a|^2 + |b|^2 = 1$, and is not the identity, then the equations

$$(a) \quad \cos \frac{\alpha}{2} = \operatorname{Re} a \quad (0 \leq \alpha < 2\pi)$$

$$\begin{aligned}
 (b) \quad \theta &= \arg b - \frac{\pi}{2} \quad (0 \leq \theta < 2\pi) \\
 \sin \phi &= \frac{|b|}{\sqrt{(1 - (\operatorname{Re} a)^2)}}, \\
 (c) \quad \cos \phi &= \frac{\operatorname{Im} a}{\sqrt{(1 - (\operatorname{Re} a)^2)}} \quad (0 \leq \phi \leq \pi)
 \end{aligned}$$

admit unique solutions in the indicated ranges. These numbers are the parameters of a rotation $\tilde{T}$ whose corresponding element in $\mathfrak{L}$ is T and hence T is in $\mathfrak{R}$.

We have shown that T is in $\mathfrak{R}$ if and only if it has the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1,$$

by a method which yields at the same time the parameters of the corresponding rotation $\tilde{T}$.

3. The theorem. If $\mathfrak{G} = (T_0, T_1, \dots)$ is any subgroup of $\mathfrak{L}$, and U any element of $\mathfrak{L}$, then $\mathfrak{G}' = (UT_0U^{-1}, UT_1U^{-1}, \dots)$ is another subgroup, isomorphic to $\mathfrak{G}$. (Indeed, we may think of $\mathfrak{G}'$ as obtained from $\mathfrak{G}$ by the (conformal) change of coordinates $z \rightarrow u(z)$.) Any group $U\mathfrak{G}U^{-1}$ is a *conjugate* of the group $\mathfrak{G}$.

THEOREM. Let $\mathfrak{G}$ be a subgroup of the linear fractional group $\mathfrak{L}$ such that $\mathfrak{G}$ contains only elliptic transformations. Then some group $\mathfrak{G}' = U\mathfrak{G}U^{-1}$, conjugate to $\mathfrak{G}$ by an element U of $\mathfrak{L}$, is contained in the subgroup $\mathfrak{R}$ of $\mathfrak{L}$, induced by rotations of the unit sphere.

4. The proof. Let there be given a group $\mathfrak{G}$ of elliptic transformations. If $\mathfrak{G}$ contains only the identity transformation

$$I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the assertion becomes trivial. Assume that $\mathfrak{G}$ contains some $T \neq I$. Then the quadratic equation $z = T(z)$ has distinct solutions z_1 and z_2 , the fixed points of T . Now the conditions $U(0) = z_1$ and $U(\infty) = z_2$ have a solution U in $\mathfrak{L}$ (in fact we still have left one degree of freedom in the choice of U). Therefore the element $U^{-1}TU$ of $U^{-1}\mathfrak{G}U$ has fixed points 0 and ∞ . After replacing $\mathfrak{G}$ by its conjugate $U^{-1}\mathfrak{G}U$ we may assume that $\mathfrak{G}$ contains some $T \neq I$ with fixed points 0 and ∞ .

Let $\mathfrak{G}_0$ be the subgroup of $\mathfrak{G}$ consisting of all T in $\mathfrak{G}$ with 0 and ∞ as fixed points. The equations $T(0) = 0$ and $T(\infty) = \infty$ imply T is diagonal,

$$T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

The condition $ad - bc = 1$ gives $d = a^{-1}$. Since T is elliptic, $|a| = 1$, whence $d = \bar{a}$. Thus the elements of $\mathfrak{G}_0$, of the form

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix},$$

with $|a| = 1$, belong to the group $\mathfrak{R}$ of all T of the form

$$T = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

We show next that every T in $\mathfrak{G}$ has the form

$$T = \begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and choose

$$S = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

in $\mathfrak{G}_0$, with $S \neq I$, hence with $\lambda \neq \pm 1$. Now

$$ST = \begin{pmatrix} \lambda a & \lambda b \\ \bar{\lambda} c & \bar{\lambda} d \end{pmatrix},$$

and the two elliptic elements T and ST of $\mathfrak{G}$ must have real traces, $a + d$ and $\lambda a + \bar{\lambda} d$. It follows first that $d = \bar{a} + h$, h real, next that $\bar{\lambda} h$ is real, whence $h = 0$ and $d = \bar{a}$.

We show that, for T in $\mathfrak{G}$, $b = 0$ iff $c = 0$. Suppose $c = 0$, hence

$$T = \begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix},$$

and take S as before. Computation shows that, since $a\bar{a} = 1$,

$$STS^{-1} = \begin{pmatrix} a & \lambda^2 b \\ 0 & \bar{a} \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} \bar{a} & -b \\ 0 & a \end{pmatrix}.$$

Then

$$K = STS^{-1}T^{-1} = \begin{pmatrix} 1 & (\lambda^2 - 1)b \\ 0 & 1 \end{pmatrix}.$$

Since K , in $\mathfrak{G}$, is elliptic and has eigenvalue 1, we must have $K = I$, that is, $(\lambda^2 - 1)b = 0$, and therefore $b = 0$. The proof that $b = 0$ implies $c = 0$ is similar.

We now show that there exists a real number $r \neq 0$ such that every T in $\mathfrak{G}$ has the form

$$\begin{pmatrix} a & b \\ r\bar{b} & \bar{a} \end{pmatrix}.$$

For this it suffices to show that if $\mathfrak{G}$ contains elements

$$T_i = \begin{pmatrix} a_i & b_i \\ r_i\bar{b}_i & \bar{a}_i \end{pmatrix}, \quad i = 1, 2,$$

with $b_i \neq 0$, $i = 1, 2$, then r_1 and r_2 are real and equal. Reality of r_i follows from the condition that $ad - bc = a\bar{a} - r\bar{b}\bar{b} = 1$. We compute the diagonal elements of the product:

$$T_1 T_2 = \begin{pmatrix} a_1 a_2 + r_2 b_1 \bar{b}_2 & * \\ * & r_1 \bar{b}_1 b_2 + \bar{a}_1 \bar{a}_2 \end{pmatrix}.$$

These elements must be complex conjugates, whence $r_2 b_1 \bar{b}_2$ is conjugate to $r_1 \bar{b}_1 b_2$, $r_2 b_1 \bar{b}_2 = r_1 \bar{b}_1 b_2$, and $r_2 = r_1$.

We show that we may choose $r = \pm 1$. We have that every T in $\mathfrak{G}$ has the form

$$\begin{pmatrix} a & b \\ r\bar{b} & \bar{a} \end{pmatrix}$$

with real $r \neq 0$. Conjugating $\mathfrak{G}$ by

$$V = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix},$$

in $\mathfrak{L}$, carries each T into

$$VTV^{-1} = \begin{pmatrix} a & v^2 b \\ v^{-2} r \bar{b} & \bar{a} \end{pmatrix}$$

with

$$r' = (v^{-2} r \bar{b}) / (\overline{v^2 b}) = r / (v \bar{v})^2.$$

Choosing v such that $|v|^4 = |r|$ gives $r' = \pm 1$. (Here we have ex-

ploited our remaining freedom in the choice of U .)

We show that $r=1$ is impossible unless each T in $\mathfrak{g}$ has $b=c=0$. Suppose then that $r=+1$ and $\mathfrak{g}$ contains

$$T = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with $b \neq 0$. If

$$S = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \quad \lambda \neq \pm 1,$$

then

$$STS^{-1} = \begin{pmatrix} a & \lambda^2 b \\ \bar{\lambda}^2 \bar{b} & \bar{a} \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix},$$

and

$$K = STS^{-1}T^{-1} = \begin{pmatrix} a\bar{a} - \lambda^2 b\bar{b} & * \\ * & * \end{pmatrix}.$$

Now the trace of K is twice the real part of $a\bar{a} - \lambda^2 b\bar{b}$, that is, twice $a\bar{a} + tb\bar{b}$, where t is the real part of $-\lambda^2$, whence $a\bar{a} + tb\bar{b} \leq 1$. But this is impossible, since T has determinant $a\bar{a} - b\bar{b} = 1$, while $-1 < t < +1$.

We have shown that we may always take $r = -1$. Thus G consists solely of transformations of the form

$$T = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

and $\mathfrak{g}$ is a subgroup of $\mathfrak{R}$, as required.

REFERENCE

1. P. Fatou, *Fonctions automorphes*, Volume II of *Théorie des fonctions algébriques*, by P. Appel and E. Goursat, Gauthier-Villars, Paris, 1930.

UNIVERSITY OF MICHIGAN