A VERSION OF THE LÉVY-BAXTER THEOREM FOR THE INCREMENTS OF BROWNIAN MOTION OF SEVERAL PARAMETERS¹

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1. Introduction. Let $X(t_1, \dots, t_k)$, $-\infty < t_1, \dots, t_k < \infty$, be Lévy's Brownian motion process of k parameters: it is a Gaussian process with mean 0 and covariance function

$$(1.1) \quad r(s, t) = EX(s_1, \dots, s_k)X(t_1, \dots, t_k) = \frac{1}{2}[\|s\| + \|t\| - \|t - s\|]$$

where $t = (t_1, \dots, t_k)$ and $\|t\| = (t_1^2 + \dots + t_k^2)^{1/2}$. For each integer $n \geq 1$, the unit cube $\{t: 0 \leq t_1 \leq 1, \dots, 0 \leq t_k \leq 1\}$ can be broken up into 2^{nk} cubes whose edges have the common length 2^{-n} . Such cubes have corner-points of the form $(i_1 2^{-n}, \dots, i_k 2^{-n})$, where the i 's are integers between 0 and 2^n . Put $i = (i_1, \dots, i_k)$. Let us denote by $Y_{i,n}$ the k th-order difference of the sample function X over the cube $\{t: (i_1 - 1)2^{-n} \leq t_1 \leq i_1 2^{-n}, \dots, (i_k - 1)2^{-n} \leq t_k \leq i_k 2^{-n}\} = C(i, n)$:

$$Y_{i,n} = \Delta_1 \cdots \Delta_k X = X(i_1 2^{-n}, \dots, i_k 2^{-n}) - \sum_{r=1}^k p_r + \sum_{r < s} p_{rs} \\ - \cdots + (-1)^k X((i_1 - 1)2^{-n}, \dots, (i_k - 1)2^{-n})$$

where $p_{rs} \dots i$ denotes $X(c_1, \dots, c_k)$ for $c_r = (i_r - 1)2^{-n}$,

$$c_s = (i_s - 1)2^{-n}, \dots, c_t = (i_t - 1)2^{-n}$$

and the remaining c_j equal $i_j 2^{-n}$.

The result we shall prove is

THEOREM. For $n \geq 1$, let $\sum |Y_{i,n}|^{2k}$ be the sum of the $2k$ th powers of the $Y_{i,n}$ over all cubes $C(i, n)$. Its limit, for $n \rightarrow \infty$, exists with probability 1 and is equal to a numerical constant B_k .

This represents a generalization of a classical theorem on the increments of the Brownian motion process of a one-dimensional time parameter, due to Lévy [2]. One can also extend this to a generalization of Baxter's theorem [1] to more general Gaussian processes of several parameters: indeed the proof of our theorem depends on the explicit form of the covariance only through the estimate in Lemma

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2.2 and the fact that the variance of $Y_{i,n}$ is bounded by a constant multiple of 2^{-n} .

2. Preliminary lemmas.

LEMMA 2.1. *Let X and Y be random variables with a joint Gaussian distribution having means 0, common variance σ^2 , and correlation coefficient ρ ; then*

$$E\{X^{2k}Y^{2k}\} - E\{X^{2k}\}E\{Y^{2k}\} \leq K\rho^2\sigma^{4k}$$

where K is some numerical constant depending only on k .

PROOF. It is sufficient to prove the inequality for the special case $\sigma=1$, as the general case follows from it. Suppose that a random variable U has a Gaussian distribution with mean m and variance s^2 . An elementary computation shows that

$$EU^{2k} = \sum_{j=0}^k c_j m^{2j} s^{2(k-j)},$$

for some constants $c_0, c_1, \dots, c_k$ with $c_0 = (2k)!/k!2^k$. The conditional distribution of X given Y is that of U for $m = \rho Y$, $s^2 = 1 - \rho^2$; therefore,

$$E[X^{2k} | Y] = \sum_{j=0}^k c_j \rho^{2j} Y^{2j} (1 - \rho^2)^{2(k-j)}.$$

Multiply each side of this equation by Y^{2k} and take expectations:

$$\begin{aligned} E[X^{2k}Y^{2k}] &= \sum_{j=0}^k c_j \rho^{2j} (1 - \rho^2)^{2(k-j)} \frac{[2(k+j)]!}{(k+j)!2^{k+j}} \\ &= [(2k)!/k!2^k]^2 \{1 - 2k\rho^2 + P(\rho)\} \end{aligned}$$

where $P(x)$ is a polynomial whose terms of degrees 0 and 1 have coefficients 0. The assertion of the lemma now follows from the inequality $|\rho|^k \leq \rho^2$ for $|\rho| \leq 1$ and $k \geq 2$.

LEMMA 2.2. *Put $r = (x_1^2 + \dots + x_k^2)^{1/2}$; then*

$$\left| \frac{\partial^{2k} r}{\partial x_1^2 \dots \partial x_k^2} \right| \leq \frac{\text{constant}}{r^{2k-1}}$$

for all values of $x_1, \dots, x_k$.

PROOF. There are constants $c_{0k}, \dots, c_{kk}$ such that

$$\frac{\partial^{2k} r}{\partial x_1^2 \dots \partial x_k^2} = \sum_{j=0}^k c_{jk} r^{-[2(k+j)-1]} \sum_{1 \leq i_1 < \dots < i_j \leq k} x_{i_1}^2 \dots x_{i_j}^2,$$

where the summation over $i_1, \dots, i_j$ is understood to be 1 for $j=0$. This expression can be verified by induction on k ; for this purpose, we note that derivatives of r with respect to $x_1, \dots, x_j$ depend on $x_{j+1}, \dots, x_k$ only through r . It follows that the derivative on the left-hand side of the above equation is dominated by a constant multiple of

$$r^{-(2k-1)} \sum_{j=0}^k \sum_{1 \leq i_1 < \dots < i_j \leq k} r^{-2j} x_{i_1}^2 \dots x_{i_j}^2.$$

The sum in the latter expression is bounded: for, on one hand, we have

$$r^{-2j} x_{i_1}^2 \dots x_{i_j}^2 \leq \frac{(x_{i_1} \dots x_{i_j})^2}{(x_{i_1}^2 + \dots + x_{i_j}^2)} j;$$

and, on the other hand,

$$x_{i_1}^2 \dots x_{i_j}^2 \leq j^{-j} (x_{i_1}^2 + \dots + x_{i_j}^2)^j$$

because the geometric mean never exceeds the arithmetic mean.

3. Proof of the Theorem. For $n \geq 1$, the random variables $\{Y_{i,n}\}$ have a joint Gaussian distribution. The means are all 0. Let us denote by D_k the variance of the k th-order difference of $X(\cdot)$ over the corner-points of the unit cube, i.e., the variance of $X(1, \dots, 1) - X(0, 1, \dots, 1) - \dots - X(1, \dots, 1, 0) + X(0, 0; 1, \dots, 1) + \dots \pm X(0, 0, \dots, 0)$. The $Y_{i,n}$ have a common variance equal to $2^{-n}D_k$: for, on one hand, the joint distribution of any finite collection of differences of the process X is invariant under translations of the parameter set, by virtue of the form (1.1) of the covariance function, and so $Y_{i,n}$ has the same distribution as $Y_{0,n}$; and, on the other hand, $Y_{0,n}$ has the variance $2^{-n}D_k$ because the process $X(ct_1, \dots, ct_k)$ is stochastically equivalent to the process $c^{1/2}X(t_1, \dots, t_k)$, for any constant $c > 0$. The covariance of $Y_{i,n}$ and $Y_{j,n}$ is equal to the $2k$ th-order difference of the function $-\frac{1}{2}\|s-t\|$ over the product of the cubes $C(i, n)$ and $C(j, n)$. If the latter cubes are disjoint, then the difference is representable as the integral

$$(3.1) \quad -\frac{1}{2} \int_{C(i,n)} \int_{C(j,n)} \frac{\partial^{2k} \|s-t\|}{\partial s_1 \dots \partial s_k \partial t_1 \dots \partial t_k} ds_1 \dots ds_k dt_1 \dots dt_k.$$

For proving the theorem it suffices to show that

$$E \left\{ \sum_i Y_{i,n}^{2k} \right\} = \frac{2k!}{k!2^k} D_k^k = B_k;$$

$$\sum_n \text{Variance} \left(\sum_i |Y_{i,n}|^{2k} \right) < \infty.$$

The first relation is directly deducible from the distribution of $Y_{i,n}$; we shall shortly verify the second relation by showing that the variance of $\sum_i |Y_{i,n}|^{2k}$ is dominated by a constant multiple of 2^{-n} , $n=1, 2, \dots$.

In the rest of the paper we omit the subscript n from $Y_{i,n}$, writing it as Y_i . The variance of $\sum_i |Y_i|^{2k}$ is

$$(3.2) \quad \sum_{ij} \{E[Y_i^{2k} Y_j^{2k}] - E[Y_i^{2k}]E[Y_j^{2k}]\}.$$

There are 2^{2nk} terms in this sum. Each term is dominated by a constant multiple of 2^{-2nk} ; in fact,

$$E[Y_i^{2k} Y_j^{2k}] \leq E[Y_i^{4k}] = 2^{-2nk} D_k^{2k} [(4k)!/(2k)!] 2^{-2k}.$$

There are at most $2^{2n(k-1)}$ terms in the sum for which the indices are restricted by an equation of the form $i_1 - j_1 = \alpha_1$ for some integer α_1 . The total contribution of such terms to the sum cannot exceed a constant multiple of 2^{-n} . The same is true of the total contribution of all terms whose indices satisfy at least one of the inequalities $|i_1 - j_1| \leq 1, \dots, |i_k - j_k| \leq 1$. We complete the proof by showing that the contribution of all terms whose indices satisfy all of the inequalities

$$(3.3) \quad |i_1 - j_1| > 1, \dots, |i_k - j_k| > 1$$

does not exceed a constant multiple of 2^{-nk} .

Lemma 2.1 implies that the general term of the sum (3.2) is bounded by a constant multiple of

$$2^{-2nk} \cdot [\text{correlation}(Y_i Y_j)]^2 = 2^{-2n(k-1)} [\text{covariance}(Y_i Y_j)]^2 \cdot \text{constant}.$$

Lemma 2.2 and equation (3.1) imply that the covariance of $(Y_i Y_j)$ is dominated by a constant multiple of

$$2^{-2nk} \max\{ \|\mathbf{s} - \mathbf{t}\|^{-(2k-1)} : \mathbf{s} \in C(i, n), \mathbf{t} \in C(j, n) \} \\ = 2^{-n} [(|i_1 - j_1| - 1)^2 + \dots + (|i_k - j_k| - 1)^2]^{-(2k-1)/2};$$

thus, the general term of (3.2) whose indices satisfy (3.3) is dominated by a constant multiple of

$$2^{-2nk} [(|i_1 - j_1| - 1)^2 + \dots + (|i_k - j_k| - 1)^2]^{-(2k-1)}.$$

The sum of all such terms is not greater than

$$2^{-2nk} \sum_{i_1, \dots, i_k=1}^{2^n} (2^n - 1)^k [i_1^2 + \dots + i_k^2]^{-(2k-1)} \\ \leq 2^{-nk} \sum_{i_1, \dots, i_k=1}^{2^n} [i_1^2 + \dots + i_k^2]^{-(2k-1)}.$$

This is dominated by

$$2^{-nk} k^{-(2k-1)} \left(\sum_{\alpha=1}^{2^n} \alpha^{-4+(2/k)} \right)^k$$

because the geometric mean of $i_1^2, \dots, i_k^2$ does not exceed the arithmetic mean. The coefficient of 2^{-nk} in the above expression is bounded. The proof is complete.

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REFERENCES

1. G. Baxter, *A strong limit theorem for Gaussian processes*, Proc. Amer. Math. Soc. **7** (1956), 522-525.
2. P. Lévy, *Le mouvement Brownien plan*, Amer. J. Math. **62** (1940), 487-550.

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