A NOTE ON A CELLULARITY THEOREM BY DOYLE

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A continuum K in S^n is cellular if $S^n \setminus K$ is topologically E^n . In [2] Doyle gave a sufficient condition for cellularity of an arc in S^n . In this paper we show that this condition is sufficient for cellularity of dendrites and can be used to get a condition for cellularity of unions of dendrites with cellular sets in S^n .

A continuum K in S^n is said to be $\mathcal{O}$ -shrinkable if there is a point q of K and for each open set U containing q there is a closed n -cell $C \subset U$ such that q lies in $\text{Int } C$ and K meets $\text{Bd } C$ in exactly one point. If y is a point of K different from q , we will say that K is $\mathcal{O}$ -shrinkable towards y . The proofs of the following lemma and theorem are identical to those given by Doyle for Lemma 1 and Theorem 1 in [2] and, therefore, will not be given here.

LEMMA 1. *Let C be a closed n -cell and N a continuum which lies in $\text{Int } C$ except for a point x of N which lies on $\text{Bd } C$. Then there is a pseudo-isotopy of C onto C which is fixed on $\text{Bd } C$ and which carries N onto x .*

THEOREM 1. *Let K be a continuum in S^n which contains a cellular subset L . Suppose that each subcontinuum K' of K which contains L as a proper subcontinuum is $\mathcal{O}$ -shrinkable. Then, K is cellular.*

COROLLARY 1. *If each subcontinuum of a dendrite in S^n is $\mathcal{O}$ -shrinkable, then the dendrite, as well as each of its subcontinua, is cellular.*

COROLLARY 2. *Suppose K is a dendrite, $y \in K$, and L is a cellular set in S^n . Suppose also that $K \cap L = \{y\}$ and that each subcontinuum of K containing y is $\mathcal{O}$ -shrinkable towards y . Then, $K \cup L$ is cellular.*

It is worthwhile to point out here that it is not enough to require that each arc of a dendrite be $\mathcal{O}$ -shrinkable in order to get cellularity of the dendrite. If we consider a dendrite on the Alexander horned sphere whose end points are the "bad" points of the horned sphere, we get a noncellular dendrite, each of whose arcs is tame and, therefore, $\mathcal{O}$ -shrinkable. If the dendrite has only a countable number of end points, then it would be sufficient to have each arc $\mathcal{O}$ -shrinkable.

Using the concept of being $\mathcal{O}$ -shrinkable, we get a local condition which gives cellularity. This condition, however, is not necessary.

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There are cellular arcs in S^3 which pierce no disk. Consider any arc of the simple closed curve described in [1]. Each subarc of such an arc would fail to be $\mathcal{P}$ -shrinkable.

To see that an arc of this simple closed curve is cellular, let $\{T_i\}_{i=1}^\infty$ be the nested sequence of solid tori which has the simple closed curve as its intersection. From T_1 remove a thickened disk C such that $Cl(T_1 \setminus C) = C_1$ is a 3-cell. Let C_2 be the 3-cell of $T_2 \cap C_1$ of largest diameter. In general let C_{n+1} be the 3-cell of $T_{n+1} \cap C_n$ of largest diameter. Let $\alpha = \bigcap_{i=1}^\infty C_i$. Since each C_i is tame, α is cellular.

REFERENCES

1. R. H. Bing, *A simple closed curve that pierces no disk*, J. Math. Pures Appl. **35** (1956), 337-343.
2. P. H. Doyle, *A sufficient condition that an arc in S^n be cellular*, Pacific J. Math. **14** (1964), 501-503.

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