ON THE STRONG (L) SUMMABILITY OF FOURIER SERIES

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1. Let $f \in L(-\pi, \pi)$ and be defined outside this range by periodicity with a period 2π . Let the Fourier series associated with the function $f(x)$ be

$$(1.1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2}A_0 + \sum_1^{\infty} A_n(x).$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}.$$

DEFINITION. A series $\sum_1^{\infty} u_r$, with partial sums s_r , is said to be strongly summable (L) with index q (>0) or summable $[L, q]$ to the sum s , provided that

$$\sum_{\nu=1}^{\infty} \frac{|s_{\nu} - s|^q}{\nu} x^{\nu} = o \left\{ \log \frac{1}{(1-x)} \right\}$$

as $x \rightarrow 1-0$.

Putting $q=1$, we get an analogue for strong summability of (L) summability $[1]$ method.

2. We prove the following theorem:

THEOREM. If $f \in L^2$, and for a fixed δ , as $t \rightarrow 0$

$$(2.1.1) \quad \int_t^{\delta} \frac{\{\phi(u)\}^2}{u} du = o \left(\log \frac{1}{t} \right)$$

then

$$(2.1.2) \quad \sum_{\nu=1}^{\infty} \frac{|s_{\nu} - s|^2}{\nu} x^{\nu} = o \left(\log \frac{1}{1-x} \right)$$

as $x \rightarrow 1-0$, and a fortiori

$$(2.1.3) \quad \sum_{\nu=1}^{\infty} \frac{|s_{\nu} - s|}{\nu} x^{\nu} = o \left(\log \frac{1}{1-x} \right).$$

The summability $[L, 2]$ implies the summability $[L, 1]$, for, from Cauchy's inequality we have

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$$\begin{aligned}
\sum_{\nu=1}^{\infty} \frac{|s_{\nu} - s|}{\nu} x^{\nu} &= \sum_{\nu=1}^{\infty} \left(\frac{|s_{\nu} - s|}{\sqrt{\nu}} x^{\nu/2} \right) \left(\frac{x^{\nu/2}}{\sqrt{\nu}} \right) \\
&\leq \left(\sum_{\nu=1}^{\infty} \frac{|s_{\nu} - s|^2}{\nu} x^{\nu} \right)^{1/2} \left(\sum_{\nu=1}^{\infty} \frac{x^{\nu}}{\nu} \right)^{1/2} \\
&= o \left(\log \frac{1}{1-x} \right) \text{ as } x \rightarrow 1-0.
\end{aligned}$$

3. We require the following lemmas:

LEMMA 1. *If (2.1.1) holds then*

$$(2.3.1) \quad \int_0^t |\phi(u)| du = o \left(t \left(\log \frac{1}{t} \right)^{1/2} \right).$$

It can be easily shown that if (2.1.1) holds; then

$$\int_0^t |\phi(u)|^2 du = o \left(t \log \frac{1}{t} \right).$$

Hence by Schwarz's inequality we have

$$\begin{aligned}
\int_0^t |\phi(u)| du &\leq \left(\int_0^t |\phi(u)|^2 du \right)^{1/2} \left(\int_0^t du \right)^{1/2} \\
&= o \left(t \left(\log \frac{1}{t} \right)^{1/2} \right).
\end{aligned}$$

LEMMA 2. *If $0 \leq t \leq \delta \leq \pi/2$, $0 \leq u \leq \delta \leq \pi/2$, $t \neq u$ then*

$$(2.3.2) \quad \sum_{\nu=1}^{\infty} \frac{\sin \nu t \sin \nu u}{\nu} x^{\nu} \leq \frac{1}{2} \log \frac{t+u}{|t-u|}.$$

We have

$$\begin{aligned}
\sum_{\nu=1}^{\infty} \frac{\sin \nu t \sin \nu u}{\nu} x^{\nu} &= \frac{1}{2} \sum_{\nu=1}^{\infty} \frac{\cos \nu |t-u| - \cos \nu(t+u)}{\nu} x^{\nu} \\
&= \frac{1}{2} \int_{|t-u|}^{t+u} \left(\sum_{\nu=1}^{\infty} \sin \nu \theta x^{\nu} \right) d\theta \\
&= \frac{1}{2} \int_{|t-u|}^{t+u} \frac{x \sin \theta}{1 - 2x \cos \theta + x^2} d\theta \\
&\leq \frac{1}{2} \log \frac{t+u}{|t-u|}
\end{aligned}$$

for

$$\frac{x \sin \theta}{1 - 2x \cos \theta + x^2} \leq \frac{1}{\theta}, \quad (0 \leq \theta \leq 2\delta \leq \pi).$$

LEMMA 3.

$$\int_1^\infty \frac{1}{t} \log \left(\frac{t+1}{t-1} \right) dt = \frac{\pi^2}{4}.$$

It is well known that

$$(2.3.3) \quad \int_0^\infty \frac{1}{t} \log \frac{t+1}{|t-1|} dt = \frac{\pi^2}{2}$$

and

$$(2.3.4) \quad \int_0^1 \frac{1}{t} \log \frac{t+1}{1-t} dt = \frac{\pi^2}{4}$$

hence the lemma follows from (2.3.3) and (2.3.4).

3. Proof of the theorem. Making the standard simplifications we may suppose without loss of generality that $f(x)$ is even, that

$$f(x) \sim \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx,$$

that the particular value of x to be considered is zero and that the sum of the series for $x = 0$ is (if it exists) also 0. So that $\phi(u) = f(u)$. We then have

$$\begin{aligned} S_n &= \frac{2}{\pi} \int_0^\pi \frac{f(u)}{2 \sin \frac{1}{2}u} \sin (n + \frac{1}{2})u du \\ &= \frac{2}{\pi} \int_0^\pi \frac{f(u)}{2 \tan \frac{1}{2}u} \sin n u du + \frac{1}{\pi} \int_0^\pi f(u) \cos n u du \\ &= \frac{2}{\pi} \int_0^\delta \frac{f(u)}{2 \tan \frac{1}{2}u} \sin n u du + \frac{2}{\pi} \int_\delta^\pi \frac{f(u)}{2 \tan \frac{1}{2}u} \sin n u du \\ &\quad + \frac{1}{\pi} \int_0^\pi f(u) \cos n u du. \end{aligned}$$

Since the last two integrals on the right tend to zero as $n \rightarrow \infty$, and

$$\int_0^\delta \frac{f(u)}{2 \tan u/2} \sin n u du = \int_0^\delta \frac{f(u)}{u} \sin n u du + o(1).$$

We have

$$\begin{aligned}
 S_n &= \frac{2}{\pi} \left(\int_0^{1-x} + \int_{1-x}^\delta \right) \frac{f(u)}{u} \sin nudu + o(1) \\
 (3.3.1) \quad &= \frac{2}{\pi} (\alpha_n + \beta_n) + o(1)
 \end{aligned}$$

say. We have from Lemma 1,

$$\begin{aligned}
 \sum_{\nu=1}^{\infty} \frac{|\alpha_\nu|^2}{\nu} x^\nu &= \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} \left(\int_0^{1-x} \frac{f(u)}{u} \sin \nu u du \right)^2 \\
 &\leq \sum_{\nu=1}^{\infty} \nu x^\nu \left(\int_0^{1-x} |f(u)| du \right)^2 \\
 (3.3.2) \quad &= \sum_{\nu=1}^{\infty} \nu x^\nu \left[o \left((1-x) \left(\log \frac{1}{1-x} \right)^{1/2} \right) \right]^2 \\
 &= o \left\{ (1-x)^2 \left(\log \frac{1}{1-x} \right) \left(\frac{x}{(1-x)^2} \right) \right\} \\
 &= o \left(\log \frac{1}{1-x} \right)
 \end{aligned}$$

as $x \rightarrow 1-0$. Again we have from Lemma 2

$$\begin{aligned}
 \sum_{\nu=1}^{\infty} \frac{|\beta_\nu|^2}{\nu} x^\nu &= \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} \int_{1-x}^\delta \frac{f(u)}{u} \sin \nu u du \int_{1-x}^\delta \frac{f(t)}{t} \sin \nu t dt \\
 (3.3.3) \quad &= \int_{1-x}^\delta \frac{f(u)}{u} du \int_{1-x}^\delta \frac{f(t)}{t} \left(\sum_{\nu=1}^{\infty} \frac{\sin \nu u \sin \nu t}{\nu} x^\nu \right) dt \\
 &\leq \frac{1}{2} \int_{1-x}^\delta \frac{f(u)}{u} du \int_{1-x}^\delta \frac{f(t)}{t} \log \left(\frac{t+u}{|t-u|} \right) dt.
 \end{aligned}$$

Since the integrand in (3.3.3) is symmetrical in t and u , the contribution of that portion of the domain of integration for which $t < u$ should be equal to the contribution of that portion for which $t > u$. Hence (3.3.3) is equal to

$$\begin{aligned}
 &\int_{1-x}^\delta \frac{|f(u)|}{u} du \int_u^\delta \frac{|f(t)|}{t} \log \left(\frac{t+u}{|t-u|} \right) dt \\
 (3.3.4) \quad &= \int_{1-x}^\delta \frac{|f(u)|}{u} du \int_1^{\delta/u} \frac{|f(u\omega)|}{\omega} \log \left(\frac{\omega+1}{\omega-1} \right) d\omega
 \end{aligned}$$

$$(3.3.5) \quad = \int_1^{\delta/(1-x)} \frac{1}{\omega} \log \left(\frac{\omega+1}{\omega-1} \right) d\omega \int_{1-x}^{\delta/\omega} \frac{|f(u)| |f(u\omega)|}{u} du.$$

The *interior* integral of (3.3.5) does not exceed

$$\begin{aligned} & \left(\int_{1-x}^{\delta/\omega} \frac{\{f(u)\}^2}{u} du \right)^{1/2} \left(\int_{1-x}^{\delta/\omega} \frac{\{f(u\omega)\}^2}{u} du \right)^{1/2} \\ &= \left(\int_{1-x}^{\delta/\omega} \frac{\{f(u)\}^2}{u} du \right)^{1/2} \left(\int_{\omega(1-x)}^{\delta} \frac{\{f(u)\}^2}{u} du \right)^{1/2} \\ &\leq \int_{1-x}^{\delta} \frac{\{f(u)\}^2}{u} du \\ &= o \left\{ \log \frac{1}{1-x} \right\} \end{aligned}$$

as $x \rightarrow 1-0$. Hence, the expression (3.3.4) is

$$(3.3.6) \quad o \left\{ \log \frac{1}{1-x} \int_1^{\delta/(1-x)} \frac{1}{\omega} \log \left(\frac{\omega+1}{\omega-1} \right) d\omega \right\} = o \left\{ \log \frac{1}{1-x} \right\}$$

as $x \rightarrow 1-0$, by Lemma 3. From (3.3.1), we have by Minkowski's inequality

$$\begin{aligned} \left(\sum_{\nu=1}^{\infty} \frac{|S_{\nu}|^2}{\nu} x^{\nu} \right)^{1/2} &\leq \frac{2}{\pi} \left(\sum_{\nu=1}^{\infty} \frac{|\alpha_{\nu}|^2}{\nu} x^{\nu} \right)^{1/2} + \frac{2}{\pi} \left(\sum_{\nu=1}^{\infty} \frac{|\beta_{\nu}|^2}{\nu} x^{\nu} \right)^{1/2} \\ &\quad + o \left(\left(\log \frac{1}{1-x} \right)^{1/2} \right) \end{aligned}$$

and from (3.3.2) and (3.3.6) we have

$$\sum_{\nu=1}^{\infty} \frac{|S_{\nu}|^2}{\nu} x^{\nu} = o \left\{ \log \frac{1}{1-x} \right\}$$

as $x \rightarrow 1-0$.

This completes the proof of the theorem. I am much indebted to Dr. Basudeo Singh for his valuable suggestions and guidance in the preparation of this paper.

REFERENCE

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