ANOTHER GENERALIZATION OF BROUWER'S FIXED POINT THEOREM

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The theorem proved here is naturally suggested by the following observation. If D^n is a closed n -cell in euclidean n -space, E^n , with boundary S^{n-1} embedded nicely enough so that there is a retraction of the closure of $E^n - D^n$ onto S^{n-1} , then any map $f: (D^n, S^{n-1}) \rightarrow (E^n, D^n)$ has a fixed point. This is an immediate consequence of the Brouwer fixed point theorem. R. H. Bing, [1], has shown that there is always a retraction, r , of the closure of the unbounded component of $E^n - S^{n-1}$ onto S^{n-1} , no matter how badly S^{n-1} is imbedded in E^n . Hence the Brouwer fixed point theorem may be generalized by applying the Lefschetz fixed point theorem to the map $\bar{r} \cdot f$, where $f: (Y, S^{n-1}) \rightarrow (E^n, Y)$, Y is the closure of the bounded component of $E^n - S^{n-1}$, $\bar{r}: (E^n, Y) \rightarrow (Y, Y)$ is defined by $\bar{r}|Y = 1$, and $\bar{r}|Cl(E^n - Y) = r$ is Bing's retraction onto S^{n-1} . Then observe that a fixed point of $\bar{r}f$ is a fixed point of f . In fact, Bing has a more general retraction theorem which suggests the following extension of the Brouwer fixed point theorem.

THEOREM. *Let X be an $(n-2)$ -connected, compact ANR which is contained in a compact, metrizable n -manifold, M , which is a rational homology n -sphere. If E is a component of $M - X$ and $p \in E$, then each map $f: (M - E, X) \rightarrow (M - p, M - E)$ has a fixed point.*

PROOF. By Theorem 3 of [1], there is a retraction, r , of $(E \cup X) - p$ onto X . Define a new map $r^*: M - p \rightarrow M - E$ by setting $r^*(x) = r(x)$ for $x \in (E \cup X) - p$, and $r^*(x) = x$ for $x \in M - E$. Suppose that $r^*f(x) = x$ for some point in $M - E$. If $f(x) \in E$, then $x = r^*f(x) \in X$, which is impossible since $f(X) \subset M - E$. Therefore $f(x) \in M - E$, and $r^*f(x) = f(x) = x$. Thus the theorem will be proved if we can show that each map of $M - E$ into itself has a fixed point.

Since M is a metrizable manifold, it is an ANR ([4, p. 98]); thus $M - E$, being a retract of $M - p$, is an ANR ([4, p. 97]). The Lefschetz fixed point theorem applies to ANR's (see [2, p. 322]) and therefore it suffices to show that $M - E$ is acyclic ([2, p. 326]). We shall use Čech and singular homology theories with rational coefficients, Q . (These theories coincide for compact ANR's ([4, p. 145]).) Let N be a homologically trivial ANR such that $M - E \subset N \subset M$. (For instance,

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let $M - N$ be the interior of a small n -cell containing p .) Since X is $(n-2)$ -connected, $H_i(X) = 0$ for $0 < i \leq n-2$ and $H_0(X) \cong Q$ ([5, p. 349]). In Čech homology theory on the category of compact pairs every triad is a proper triad ([3, p. 266]). Therefore, we may apply the Mayer-Vietoris sequence to the triad $(N, (E \cup X) \cap N, M - E)$ and conclude that $H_i(M - E) = 0$, for $0 < i \leq n-2$, and $H_0(M - E) \cong Q$. Since $M - E$ is a proper subset of a connected n -manifold, it follows that $H_i(M - E) = 0$ for $i \geq n$. (We assume here that $n > 0$. If $n = 0$, the theorem is trivial.) By Alexander duality, (see [6, p. 263]) since E is arc-wise connected, $H_{n-1}(M - E) = 0$. Therefore $M - E$ is acyclic and the theorem follows.

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A T_1 -COMPLEMENT FOR THE REALS

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The family of all topologies definable on an arbitrary set X forms a complete lattice Σ under the partial ordering: $\tau_1 \leq \tau_2$ if and only if $\tau_1 \subseteq \tau_2$. The lattice operations $\wedge$ and $\vee$ are defined as: $\tau_1 \wedge \tau_2 = \tau_1 \cap \tau_2$ and $\tau_1 \vee \tau_2$ is the topology generated by the base $\mathcal{B} = \{B: B = U_1 \cap U_2, U_1 \in \tau_1 \text{ and } U_2 \in \tau_2\}$. The greatest element, 1, is the discrete topology and the least element, 0, is the trivial topology. The lattice Σ has been recently studied [2], [3], [4] and has been shown to be complemented [4].

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