NONSINGULAR RINGS

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1. Introduction. This paper is intended to extend some well-known results for integral domains to nonsingular rings and to give a characterization of right-semihereditary rings using their maximal right quotient rings.

For homological notions the reader is referred to [2]. For the notions, singular submodule, finite dimensional module, the reader is referred to [7], [8]. For the notion of maximal quotient ring see [5].

For a module M_R , $Z(M_R)$ denotes the singular submodule.

All rings are rings with identity and all modules are unitary.

2. Semihereditary and hereditary rings. Gentile [6] has shown that if R is a ring such that every principal right ideal is projective, then $Z(R_R) = 0$ and if R is commutative and every principal ideal is R -flat, then $Z(R) = 0$, hence right (left) semihereditary rings have zero singular right (left) ideal.

THEOREM 2.1. *If R is a ring such that $Z(R_R) = 0$ and P_R is an R -projective module containing a finitely generated large submodule, then P_R is finitely generated.*

PROOF. By [2, Proposition VII, 3.1], there exists a family $\{x_\alpha\}$ of elements of P_R and a family $\{f_\alpha\} \subseteq {}_R P^* = \text{Hom}(P_R, R_R)$ such that for all $x \in P$, $x = \sum x_\alpha f_\alpha(x)$, where $f_\alpha(x) = 0$ for all but a finite number of α .

Clearly, it is sufficient to show that f_α is the zero map for all but finitely many α .

Let B_R be a finitely generated large submodule of P_R , with generators $b_1, \dots, b_n$, then $\{\alpha | f_\alpha(b_i) \neq 0 \text{ for some } 1 \leq i \leq n\} = A$ is a finite set of the indices and $f_\alpha(B) = 0$, if $\alpha \notin A$. If $x \in P$, then $I = \{r \in R | xr \in B\}$ is a large right ideal of R by [10, Proposition 1.2] and for $\alpha \notin A$, $0 = f_\alpha(xr) = f_\alpha(x)r$ for every $r \in I$, so $f_\alpha(x) \in Z(R_R) = 0$ and the theorem follows.

COROLLARY 1. *If R is a ring such that $Z(R_R) = 0$ and $d(R_R)$ is finite, then every R -projective right ideal is finitely generated.*

PROOF. If I_R is an R -projective right ideal of R , then I_R contains as a large submodule of I_R a finitely generated right ideal, by [10, Theorem 1.3], hence by the theorem I_R is finitely generated.

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COROLLARY 2. *If R is a right hereditary ring with $d(R_R)$ finite, then R is right Noetherian.*

PROOF. Since R is right hereditary $Z(R_R)=0$ by remark in the Introduction. By Corollary 1, every R -projective right ideal is finitely generated, so R is right Noetherian since every right ideal is projective.

Corollary 1 generalizes the result that over a commutative integral domain every projective ideal is finitely generated and Corollary 2 generalizes the result that a commutative integral domain which is a hereditary ring is Noetherian.

The notion of a closed submodule as considered in [8] for example will be useful.

DEFINITION 2.2. A submodule B_R of a right R -module A_R is closed in A if B has no proper essential extension in A , i.e. if C is a submodule containing B with B large in C , then $B=C$.

LEMMA 2.3. (a) *If B is a submodule of A such that $Z(A/B)=0$, then B is closed in A .*

(b) *If B is closed in A and $Z(A)=0$, then $Z(A/B)=0$.*

PROOF. (a) If C is an essential extension of B in A and $x \in C$, then $I = \{r \in R \mid xr \in B\}$ is a large right ideal of R , by [10, Proposition 1.2], hence $x+B \in Z(A/B)=0$, so $x \in B$ and it follows that B is closed in A .

(b) If $x+B \in Z(A/B)$, then $I = \{r \in R \mid xr \in B\}$ is a large right ideal of R . The claim is that $B+xR$ is an essential extension of B . If $y=b+xt \in B+xR$ with $b \in B, t \in R$, then $J = \{r \in R \mid tr \in I\}$ is a large right ideal of R . Since $Z(A)=0$, there is an $r \in J$ such that $yr \neq 0$, but $yr \in B$, so B is a large submodule of $B+xR$. Since B is closed, $x \in B$, so $x+B=0$ and (b) follows.

The proofs of the results in the following proposition can be found in [8].

PROPOSITION 2.4. (a) *If M_R is a finite dimensional right R -module and K_R is a closed submodule of M , then M/K is finite dimensional and $d(M)=d(M/K)+d(K)$ (where $d(M)=\dim(M)$ of [8]).*

(b) *If $M_R=M_1 \oplus \cdots \oplus M_n$ is the direct sum of the finite dimensional submodules $M_j, j=1, 2, \dots, n$, then M is finite dimensional and $d(M)=\sum_{j=1}^n d(M_j)$.*

LEMMA 2.5. *If R is a finite dimensional right R -module and M_R is a finitely generated nonsingular right R -module, then M_R is finite dimensional.*

PROOF. M is an epimorphic image of a finitely generated free right

R -module F_R with kernel K_R . By Proposition 2.4(b), F_R is of finite dimension. Since M is isomorphic to F/K , F/K is nonsingular, so by Lemma 2.3(a), K is a closed submodule of F . Now by Proposition 2.4(a), F/K is finite dimensional, hence so is M .

THEOREM 2.6. *If R is right semihereditary and finite dimensional as a right R -module, then every finitely generated torsionless [1] right R -module is R -projective.*

PROOF. Let M_R be a finitely generated torsionless right R -module. Since R is semihereditary, $Z(R_R) = 0$, hence it is clear that $Z(M_R) = 0$, since M is contained in a direct product of copies of R_R . By Lemma 2.5, it follows that M_R is finite dimensional.

Since M is torsionless there is $0 \neq f \in {}_R M^* = \text{Hom}(M_R, R_R)$. Since $\text{Im}(f)$ is a finitely generated right ideal of R , $K = \text{Ker}(f)$ is a direct summand of M , because $\text{Im}(f)$ is R -projective. Let $M = K \oplus I$. Now proceed by induction on $d(M) = n$, dimension of M , [8].

If $n = 1$, then $K = 0$ since $f \neq 0$, so M is isomorphic to $\text{Im}(f)$, hence R projective. If $K = 0$, then M_R is R -projective. If $K \neq 0$, then K is finitely generated torsionless, and $d(K) < d(M)$ since $\text{Im}(f) \cong I \neq 0$, hence by induction, K is R -projective and the theorem follows.

COROLLARY. *If R is right semihereditary and finite dimensional as a right R -module, then R is left semihereditary.*

PROOF. In view of Chase [3, Theorem 4.1] it is sufficient to show that every torsionless right R -module is R -flat. Since submodules of torsionless modules are torsionless and the functors Tor_n^R commute with direct limits, it is sufficient to show that a finitely generated torsionless right R -module is R -flat, because a module is the direct limit of its finitely generated submodules. The corollary follows from the theorem since R -projective modules are R -flat.

THEOREM 2.7. *Let Q be a ring such that Q is right Q -injective, then every finitely generated nonsingular right Q -module is Q -projective and Q -injective.*

PROOF. Let M be a finitely generated nonsingular right Q -module. Then M is the epimorphic image of a free right Q -module F with finite basis. Let K denote the kernel of the epimorphism. Since F is free on a finite basis, F is Q -injective. Since M is nonsingular K is closed in F . Clearly a closed submodule of an injective module is a direct summand, hence it follows that M is Q -projective and Q -injective.

LEMMA 2.8. *If R is a ring with $Z(R_R) = 0$ and $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ a*

short exact sequence of right R -modules such that F is free, K contains a finitely generated large submodule L and M is R -flat, then M is R -projective.

PROOF. Let $u_1, \dots, u_n$ be generators of the finitely generated large submodule L of K . Since M is R -flat by [3, Proposition 2.2] there is an R -homomorphism $\theta: F \rightarrow K$ such that θ is the identity map on L . If $x \in K$, then $I = \{r \in R \mid xr \in L\}$ is a large right ideal of R . For $r \in I$, $xr = \theta(xr) = \theta(x)r$ so $(x - \theta(x))r = 0$ for each $r \in I$, hence $x - \theta(x) \in Z(F) = 0$ since $Z(R_R) = 0$. Thus, it is shown that θ is the identity on K , so the short exact sequence splits and the lemma follows.

THEOREM 2.8. Let R be a ring with $Z(R_R) = 0$, Q its maximal right quotient ring and M_R a finitely generated right R module. Then M_R is R -projective if and only if M_R is R -flat and $M \otimes_R Q$ is Q -projective.

PROOF. There is an exact sequence $0 \rightarrow K_R \rightarrow F_R \rightarrow M_R \rightarrow 0$ where F_R is free on a finite basis, K is the kernel of the epimorphism $F \rightarrow M$ and $K \rightarrow F$ is the inclusion map.

Only if: If M_R is R -projective, then the sequence splits and therefore the sequence

$$0 \rightarrow K \otimes_R Q \rightarrow F \otimes_R Q \rightarrow M \otimes_R Q \rightarrow 0$$

of right Q -modules is split exact. Since $F \otimes_R Q$ is a free right Q -module, it follows that $M \otimes_R Q$ is right Q -projective.

If: In view of Lemma 2.8 it is sufficient to show that K contains a finitely generated large submodule. Since M_R is R -flat and $M \otimes_R Q$ is Q -projective, the sequence $0 \rightarrow K \otimes Q \rightarrow F \otimes Q \rightarrow M \otimes Q \rightarrow 0$ is split exact and it follows that $K \otimes Q$ is a finitely generated right Q -module. Clearly a finite set of generators of $K \otimes Q$ as a right Q -module of the form $k_i \otimes 1$, $i = 1, \dots, n$ with $k_i \in K$ exists. Let L be the submodule of K generated by $k_1, \dots, k_n$. Since K is nonsingular, the sequence $0 \rightarrow K \rightarrow K \otimes Q$ is exact by [10, Proposition 2.2], K may be identified in $K \otimes Q$ with those elements of the form $x \otimes 1$, $x \in K$. If $0 \neq x \in K$, then $x \otimes 1 = \sum_{i=1}^n k_i \otimes q_i$ with $q_i \in Q$. Let $I = \{r \in R \mid q_i r \in R \text{ for all } i\}$, then by [10, Proposition 1.2], I is a large right ideal of R . Clearly, $K \otimes Q$ is nonsingular as a right R -module since Q_R is, there is an $r \in I$ such that $0 \neq xr \otimes 1 = (x \otimes 1)r = \sum_i k_i \otimes q_i r = \sum_i k_i (q_i r) \otimes 1$ and it follows that $0 \neq xr \in L$ since $\sum_i k_i (q_i r) \in L$, so L is a large submodule of K and the if part follows.

COROLLARY 1. If R is a ring with $Z(R_R) = 0$ and R_R is finite dimensional, then every finitely generated flat right R -module is R -projective.

PROOF. In this case, Q , the maximal right quotient ring of R , is

semisimple with minimum condition, by [10, Theorem 2.3], so every Q -module is Q -projective and the corollary follows.

A special case of Corollary 1 is

COROLLARY 2. *If R is a commutative integral domain, then every finitely generated R -flat R -module is projective.*

Endo [4], has also established Corollary 2 as a consequence of a more general theorem for commutative rings and their total quotient rings. However, Corollary 2 can be proven directly, without appealing to any quotient rings, using Lemma 2.8.

THEOREM 2.9. *Let R be a ring with $Z(R_R) = 0$, Q its maximal quotient ring such that ${}_R Q$ is left R -flat. Then every finitely generated R -flat submodule of a free right R -module is R -projective.*

PROOF. Let M_R be a finitely generated submodule of the free right R -module F_R . Since ${}_R Q$ is left R -flat the sequence of right Q -modules $0 \rightarrow M \otimes_R Q \rightarrow F \otimes_R Q$ is exact, hence it follows that $M \otimes_R Q$ is a nonsingular right Q -module since Q is. Clearly, $M \otimes_R Q$ is a finitely generated right Q -module and Q is right Q -injective module, by Theorem 2.7, $M \otimes_R Q$ is Q -projective. If M_R is R -flat, then by Theorem 2.8 M_R is R -projective.

It is well known, e.g. [3], that a commutative integral domain R is semihereditary (Prüfer ring) if and only if $\text{GWD}(R) \leq 1$. Chase [3] generalized this for arbitrary semihereditary rings. Another generalization will be given here.

THEOREM 2.10. *For a ring R the following statements are equivalent.*

- (1) R is right semihereditary.
- (2) $Z(R_R) = 0$, $\text{GWD}(R) \leq 1$ and Q the maximal right quotient ring of R is left R -flat.

PROOF. (1) implies (2). $Z(R_R) = 0$ by the remark at the beginning of this paper and that $\text{GWD}(R) \leq 1$ is well known, e.g. [2]. To show that ${}_R Q$ is left R -flat it is sufficient to show that if I is a finitely generated right ideal of R , then the sequence $0 \rightarrow I \otimes_R Q \rightarrow R \otimes_R Q$ is exact. Since I is R -projective, by Theorem 2.8, $I \otimes_R Q$ is Q -projective, hence nonsingular as a right R -module. If $\sum_{i=1}^n a_i \otimes q_i = 0$ in $R \otimes_R Q$, with $a_i \in I$, $q_i \in Q$, then $\sum_i a_i q_i = 0$ and $J = \{r \in R \mid q_i r \in R \text{ all } i\}$ is a large right ideal in R by [10, Proposition 1.2]. In $I \otimes_R Q$, for $r \in J$, $(\sum_i a_i \otimes q_i)r = \sum_i a_i \otimes q_i r = (\sum_i a_i (q_i r)) \otimes 1 = 0 \otimes 1 = 0$, since $\sum_i a_i q_i = 0$, so $\sum_i a_i \otimes q_i \in Z(I \otimes_R Q) = 0$ and (2) follows.

(2) implies (1). If I_R is a finitely generated right ideal of R , then I_R is R -flat since $\text{GWD}(R) \leq 1$, so by Theorem 2.9 I_R is R -projective and (1) follows.

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