ON THE RESTRICT-INDUCE MAP OF GROUP CHARACTERS

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The purpose of this note is to give a new proof of the following theorem due to E. Artin.

THEOREM. *Each ordinary irreducible character of a finite group is a linear combination with rational coefficients of characters induced from linear characters of cyclic subgroups.*

Let G be a finite group and let $K_1, \cdots, K_n$ be the classes of conjugate elements of G. Then the ordinary irreducible characters $\chi_1, \cdots, \chi_n$ form a basis of the vector space U of all complex-valued class functions on G over the complex number field. This basis is orthonormal relative to the usual inner product. There is another orthonormal basis $\alpha_1, \cdots, \alpha_n$ defined by

$$\begin{aligned} \alpha_i(g) &= 0 \quad \text{if } g \in G - K_i, \\ &= |C(g)|^{1/2} \quad \text{if } g \in K_i, \end{aligned}$$

where $|C(g)|$ is the order of the centralizer $C(g)$ of g in G.

Let H be a subgroup of G. We define a linear transformation T of U by $T(\theta) = (\theta|H)^*$, the class function on G obtained by inducing the restriction of θ to H. It will be noted that T is symmetric, for, applying Frobenius reciprocity twice, we have

$$\begin{aligned} (T(\theta), \eta)_G &= ((\theta | H)^*, \eta) = (\theta | H, \eta | H)_H \\ &= (\theta, (\eta | H)^*) = (\theta, T(\eta))_G. \end{aligned}$$

Let

$$T(\chi_i) = \sum_j a_{ij}\chi_j$$

so that $A = (a_{ij})$ is the matrix of T relative to the basis $\chi_1, \cdots, \chi_n$. It is known that each a_{ij} is a nonnegative rational integer.

If $K_i \cap H$ is empty, the formula for the value of an induced character implies $T(\alpha_i) = 0$. If $K_i \cap H$ is nonempty, then

$$K_i \cap H = C_1 + \cdots + C_s$$

where each C_j is a class of H. Let $g \in K_i$ and let $g_j \in C_{j,s}$ $j = 1, \cdots, s$. Then

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$$\begin{aligned} T(\alpha_i)(g) &= [|\,C(g)\,|\;|K_i \cap H\,|\,/\,|\,H\,|\,]\alpha_i(g) \\ &= \left[(|\,C(g)\,|\,/\,|\,H\,|\,)\sum_j |\,H:H\cap C(g_j)\,|\right]\alpha_i(g) \\ &= \left[\sum_j |\,C(g)\,|\,/\,|\,H\cap C(g_j)\,|\right]\alpha_i(g) \\ &= \left[\sum_j |\,C(g_j):C(g_j)\cap H\,|\right]\alpha_i(g). \end{aligned}$$

Since $T(\alpha_i)(g)=\alpha_i(g)=0$ for $g\in G-K_i$, we have

$$T(\alpha_i) = r_i\alpha_i$$

where $r_i=\sum_j |\,C(g_j):C(g_j)\cap H|$ is a sum of positive integers. Hence $\alpha_1,\cdots,\alpha_n$ is a basis of eigenvectors of T and the eigenvalues $r_1,\cdots,r_n$ are nonnegative integers. The above shows that the rank of T is the number of classes of G meeting H. The kernel of T is the set of class functions on G vanishing on those classes meeting H.

The two matrices of T, A and diag $\{r_1,\cdots,r_n\}$, are similar over the complex number field. Since their entries are rational numbers, they are already similar over the rational number field Q. This implies the existence of a basis $\theta_1,\cdots,\theta_n$ of eigenvectors of T which are Q-linear combinations of $\chi_1,\cdots,\chi_n$ with eigenvalues $r_1,\cdots,r_n$.

We consider $\theta_1,\cdots,\theta_n$ as a basis of the vector space $V(G)$ of all Q-linear combinations of $\chi_1,\cdots,\chi_n$ and restrict T to $V(G)$. Assume the notation is chosen so that $r_i\neq 0$, $i=1,\cdots,t$ while $r_i=0$ for $i=t+1,\cdots,n$. Note that for $i\leqq t$, $\theta_i(g)=0$ if g lies in a class of G not meeting H, because $\theta_i(g)=(1/r_i)T(\theta_i)(g)=0$. On the other hand for $i>t$, $\theta_i(g)=0$ if g lies in a class meeting H. Now let χ be an irreducible character of G and $a_1,\cdots,a_n$ rational numbers such that $\chi=\sum_{j=1}^n a_j\theta_j$. Set $\mu=\sum_{j=1}^t a_j\theta_j$. It follows that μ vanishes on those classes not meeting H and that $\mu|H=\chi|H$. Furthermore,

$$\mu = \sum_{j=1}^{t}\frac{a_j}{r_j}(r_j\theta_j) = \sum_{j=1}^{t}\frac{a_j}{r_j}(\theta_j\,|\,H)^*.$$

Since $\theta_j|H$ is a Q-linear combination of irreducible characters of H, we have the following proposition.

Proposition. *Given an irreducible character χ of G and a subgroup H of G, there exists a Q-linear combination μ of characters of G induced from irreducible characters of H such that $\mu=0$ on classes of G not meeting H and $\mu|H=\chi|H$.*

The function μ is not uniquely determined. In the following paragraph we write $\chi(H)$ for one such function, indicating its dependence on the given character χ and the given subgroup H. The particular choice of $\chi(H)$ is immaterial.

Now let $g_i \in K_i$, $i=1, 2, \cdots, n$ and let H_i be the cyclic subgroup generated by g_i. Let χ be an irreducible character of G. The proof of the main result is completed by noting

$$\begin{aligned}\chi = {} & \chi(H_1) + \chi(H_2) + \cdots + \chi(H_n) \\ & - \chi(H_1 \cap H_2) - \chi(H_1 \cap H_3) - \cdots - \chi(H_{n-1} \cap H_n) \\ & + \chi(H_1 \cap H_2 \cap H_3) + \cdots + \chi(H_{n-2} \cap H_{n-1} \cap H_n) \\ & \vdots \\ & + (-1)^{n+1}\chi(H_1 \cap H_2 \cap \cdots \cap H_n)\end{aligned}$$

where in the kth row there is one and only one term for each possible intersection of k of the H_i's. (This expression holds whenever $H_1, \cdots, H_n$ is a set of subgroups of G such that each class of G meets at least one H_i.)

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