FRACTIONAL DIFFERENTIATION OF FUNCTIONS WITH LACUNARY FOURIER SERIES¹

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1. This paper gives some results of a special nature for functions which have lacunary Fourier series. Before stating these, we begin with some definitions and comments.

Let $f: (0, 2\pi) \rightarrow \mathbb{R}$ be integrable and assume it has the Fourier series

$$(1) \quad S(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos n_k x + b_k \sin n_k x.$$

f is said to have a lacunary Fourier series if there is a $\lambda > 1$ such that for $k=1, 2, \dots$ we have $n_{k+1} > \lambda n_k$. We will assume $a_0 = 0$. Sometimes we will prefer to write $S(f)$ in its complex form

$$(1') \quad S(f) = \sum'_{-\infty}^{+\infty} C_k \exp(in_k x),$$

where $\sum'$ indicates that the sum excludes the term $k=0$.

Let $0 \leq k < \alpha < k+1$ and $\alpha + \beta = k+1$, with k an integer, be given. We define the β th integral of f by

$$(2) \quad F(x) = f_\beta(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^x f(t)(x-t)^{\beta-1} dt.$$

A discussion of this integral is given in [6, page 133]. There it is shown that

$$S(f_\beta) = \sum'_{-\infty}^{\infty} \frac{C_k}{(in_k)^\beta} \exp(in_k x),$$

and it is pointed out that the convergence of the integral is bound up with the fact that $a_0 = 0$. From this it is clear that F has a lacunary Fourier series if f does. f is said to have an α th derivative if F has a $k+1$ Peano derivative at x_0 , i.e., there exists a polynomial $P_{x_0}(t)$ of degree $\leq k+1$ such that

$$(3) \quad F(x_0 + t) - P_{x_0}(t) = o(|t|^{k+1}), \quad t \rightarrow 0.$$

If

Received by the editors November 30, 1966.

¹ This is a portion of the author's doctoral dissertation at Purdue University.

² Partially supported by NSF Grant GP-1665.

$$\left\{ \frac{1}{\rho} \int_{-\rho}^{\rho} |F(x_0 + t) - P_{x_0}(t)|^p dt \right\}^{1/p} = o(\rho^{k+1}),$$

$$\rho \rightarrow 0 \quad (1 \leq p < \infty),$$

f is said to have an α th derivative in the L^p sense. If there exists a polynomial $Q_{x_0}(t)$ of degree $\leq k$ such that $R_{x_0}(t) = f(x_0 + t) - Q_{x_0}(t) = O(|t|^\alpha)$ as $t \rightarrow 0$ then f is said to satisfy the condition Λ_α and if

$$\left\{ \frac{1}{\rho} \int_{-\rho}^{\rho} |R_{x_0}(t)|^p dt \right\}^{1/p} = O(\rho^\alpha), \quad \rho \rightarrow 0 \quad (1 \leq p < \infty),$$

f is said to satisfy the condition Λ_α^p . f is said to satisfy the condition N_α^p if for some $\rho > 0$

$$\frac{1}{\rho} \int_{-\rho}^{\rho} \frac{|R_{x_0}(t)|^p}{|t|^{1+p\alpha}} dt < \infty.$$

If $\alpha = 1$, the condition Λ_1 is replaced by the condition Λ_* , which requires that $\Delta^2 f(x_0, t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0) = O(t)$ as $t \rightarrow 0$. Finally, f is said to have an approximate derivative at x_0 , equal to $f'_{ap}(x_0)$, if $|f(x_0 + t) - f(x_0) - f'_{ap}(x_0)t| = o(t)$ as $t \rightarrow 0$ through a set having zero as a point of density.

Throughout, constants depending only on the parameters of the problem will be denoted by D_k , D , M , C etc. These will not necessarily denote the same constant at each point.

The results we will state were motivated by the main result in [4] which gives the following theorem for the case $p = 2$.

THEOREM A. *The necessary and sufficient condition for f to satisfy the conditions N_α^p , N_α^2 ($2 \leq p < \infty$) almost everywhere in a measurable set E is that f satisfy the condition Λ_α^p and have an α th derivative in the L^p sense almost everywhere in the set E .*

The proof of this theorem follows the proof of the theorem in [4] with $p = 2$ and a few minor modifications. The results in [4] parallel the results of [3] on differentiability of functions.

We are now ready to state the two theorems of this paper.

THEOREM 1. *Let f have a lacunary Fourier series and suppose f has an α th derivative at some point x_0 ($0 \leq k < \alpha < k + 1$). Then f satisfies the condition Λ_α everywhere.*

THEOREM 2. *Let F be a function with a lacunary Fourier series. If $\sum (a_j^2 + b_j^2) n_j^2 < \infty$ then F has a derivative almost everywhere.*

The first of these is proved by a method similar to one in [5] which

is adapted to higher orders of differentiation by means of Taylor's theorem. The second is a corollary of the main theorem in [3]. It is included here because of some of its corollaries.

2. We start with the proof of Theorem 1. We may assume $x_0 = 0$ and, under the assumption that the polynomial in (3) is identically zero, we have $F(x_0 + t) = F(t) = R(t)$ where $R(t) = O(|t|^{k+1})$ as $t \rightarrow 0$. Let $N = N_j = \min\{n_j - n_{j-1}, n_{j+1} - n_j\}$. We can choose an integer $p > 0$ such that $N - 1 < p(k+2) \leq N + k + 1$. Let

$$K^j(t) = K(t) = B_k^{-1} \left(\frac{\sin(pt/2)}{p \sin(t/2)} \right)^{2(k+2)},$$

where

$$B_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\sin(pt/2)}{p \sin(t/2)} \right)^{2(k+2)} dt.$$

$K(t)$ is a trigonometric polynomial of degree $(p-1)(k+2)$ and $(1/\pi) \int_{-\pi}^{\pi} K(t) dt = 1$. Hence we may write $K(t) = 1/2 + \sum_{\sigma=1}^{N-1} d_{\sigma} \cos \sigma t$ where

$$d_{\sigma} = \frac{1}{\pi} \int_{-\pi}^{\pi} K(t) \cos \sigma t dt \quad \sigma = 1, 2, \dots, N-1.$$

Because

$$\begin{aligned} B_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\sin(pt/2)}{p \sin(t/2)} \right)^{2(k+2)} dt > \frac{2}{\pi} \int_0^{1/p} \left(\frac{\sin(pt/2)}{p \sin(t/2)} \right)^{2k+4} dt \\ &\geq \frac{1}{p} \left(\frac{2}{\pi} \right)^{2k+5} = \frac{1}{p} D_k, \end{aligned}$$

it can be shown that

$$\begin{aligned} K(t) &\leq D'_k p, \quad 0 \leq |t| \leq 1/N, \\ &\leq D'_k (1/p^{k+2} |t|^{k+3}), \quad 1/N \leq |t| \leq \pi, \end{aligned}$$

where we recall $k < \alpha < k+1$ is fixed.

It is easy to show that

$$\begin{aligned} 2K(t) \cos n_j t &= \cos n_j t + \sum_{\sigma=1}^{N-1} d_{\sigma} [\cos(n_j + \sigma)t + \cos(n_j - \sigma)t], \\ 2K(t) \sin n_j t &= \sin n_j t + \sum_{\sigma=1}^{N-1} d_{\sigma} [\sin(n_j + \sigma)t + \sin(n_j - \sigma)t]. \end{aligned}$$

Since $n_{j-1} < n_j - \sigma < n_j + \sigma < n_{j+1}$ for $\sigma = 1, 2, \dots, N-1$, we have

$$(4) \quad (a_j, b_j) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) 2K(t) (\cos n_j t, \sin n_j t) dt,$$

where a_j, b_j are the Fourier coefficients of F .

Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that for $|x| < \delta$ $|F(x)| < \epsilon |x|^{k+1}$ since f has an α th derivative at $x=0$. Assuming $1/N < \delta$ and applying formula (4) we obtain

$$\begin{aligned} |a_j| &\leq \frac{2}{\pi} \int_0^{\pi} |F(x)| K(x) dx \leq D'_k N \epsilon \int_{0 < |x| \leq 1/N} |x|^{k+1} dx \\ &\quad + D'_k N^{-(k+2)} \epsilon \int_{1/N \leq |x| \leq \delta} |x|^{-2} dx \\ &\quad + D'_k N^{-(k+2)} \int_{\delta \leq |x| \leq \pi} |F(x)| \cdot |x|^{-(k+3)} dx \\ &\leq M_1 \epsilon N^{-(k+1)} + M_2 \delta^{-(k+3)} N^{-(k+2)} \int_{-\pi}^{\pi} |F(x)| dx \\ &\leq M_1 (L)^{k+1} \epsilon (n_j)^{-(k+1)} + M_3 \frac{\delta^{-(k+3)}}{n_j} (L)^{k+2} \left(\frac{1}{n_j} \right)^{k+1}, \end{aligned}$$

where M_1, M_2 and M_3 are constants and $L = \min \{ \lambda - 1, 1 - 1/\lambda \}$. Since M_1, M_2, M_3 and L are all fixed constants and we can choose j so large that $M_3 \delta^{-(k+3)} < n_j \epsilon$, we see that $a_j = o(n_j^{-(k+1)})$ as $j \rightarrow \infty$. Similarly, $b_j = o(n_j^{-(k+1)})$ as $j \rightarrow \infty$.

If we denote by $c_j(F)$ and $c_j(f)$ the Fourier coefficients of F and f respectively we see that $c_j(F) = o(n_j^{-(k+1)})$ as $|j| \rightarrow \infty$ and hence that $c_j(f) = o(n_j^{-(k+1)+\beta}) = o(n_j^{-\alpha})$ as $|j| \rightarrow \infty$. A result in [2] shows that for $0 < \alpha < 1$ we have f satisfies the condition Λ_α if and only if $c_j = O(n_j^{-\alpha})$.

Suppose that $k < \alpha < k+1$. Then the series

$$\sum_{-\infty}^{\infty} c_j (in_j)^k \exp(in_j t)$$

converges uniformly. To see this, let $\epsilon > 0$ be given. Then there exists $M > 0$ such that $|c_j| \leq M n_j^{-\alpha}$, $j = \mp 1, \mp 2, \dots$. Choose N such that for $|j| \geq N$, $|n_j|^{\beta-1} \leq C\epsilon$, where $C^{-1} = 2M \sum_{i=1}^{\infty} (\lambda^{\beta-1})^i$. Then we see that

$$\begin{aligned} \sum_{|j| \geq N} |c_j| \cdot |n_j|^k &\leq \sum_{|j| \geq N} M \cdot |n_j|^{-\alpha} \cdot |n_j|^k \\ &\leq 2M |n_N|^{\beta-1} \sum_{i=N}^{\infty} (\lambda^{\beta-1})^i \leq C^{-1} C \epsilon = \epsilon. \end{aligned}$$

This shows that f has k derivatives and that the k th derivative has coefficients which are $o(n_j^{\beta-1})$ as $j \rightarrow \infty$, and hence $f^k(x)$ satisfies the condition $\Lambda_{1-\beta}$.

We now apply Taylor's theorem with remainder to see that

$$f(x+t) = \sum_{j=1}^k f^j(x) \frac{t^j}{j!} + R(t)$$

where

$$R(t) = \frac{1}{(k-1)!} \int_0^t (t-u)^{k-1} [f^k(x+u) - f^k(u)] du.$$

Since $f^k(x)$ satisfies $\Lambda_{1-\beta}$, it follows that $f(x)$ is in Λ_α because

$$R(t) = O\left(\int_0^t (t-u)^{k-1} u^{1-\beta} du\right) = O(|t|^{k+1-\beta}) = O(|t|^\alpha), \quad t \rightarrow 0.$$

Finally, suppose that the polynomial in (3) is not identically zero, then we can find a C_0^∞ function g which has this polynomial as its Taylor's expansion up to the $(k+1)$ th term at the point $x=0$ and with support in $(-\pi, \pi)$. It is easy to see that g has Fourier coefficients which are $O(n^{-(k+3)})$. Let $G=F-g$ and let F be replaced by G in the integral formula (4). Then $|a_j|$ is dominated by that integral $+O(\sum_{n_j^{j+1}} |a_j(g)|)$ where $a_j(g)$ are the Fourier coefficients of g . The integral is $O(n_j^{-(k+1)})$ as before and

$$\begin{aligned} \sum_{n_j}^{n_{j+1}} |a_j(g)| &= O\left(\sum_{n_j}^{\infty} n_j^{-(k+3)}\right) = O\left(n_j^{-(k+1)} \sum_{n_j} n^{-2}\right) \\ &= o(n_j^{-(k+1)}), \quad j \rightarrow \infty. \end{aligned}$$

Hence $|a_j| = o(n_j^{-(k+1)})$. This completes the proof of Theorem 1.

3. To begin the proof of Theorem 2 we note that because of the main result in [3] we only need to show that f satisfies the condition Λ_* and that the integral

$$(5) \quad \int_{-\pi}^{\pi} \frac{|\Delta^2 f(x, t)|^2}{|t|^3} dt < \infty \quad \text{for almost all } x.$$

However, since $a_j, b_j = O(n_j)$ as $j \rightarrow \infty$, it can be shown using the technique of [6, Volume I, page 47] that f satisfies the condition Λ_* .

We show the integral (5) is finite for almost every x in $[-\pi, \pi]$ by showing the following integral is finite

$$(6) \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|\Delta^2 f(x, t)|^2}{|t|^3} dt dx.$$

This is equal to

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{|t|^3} \left(- \sum [a_k \cos n_k x + b_k \sin n_k x] 4 \sin^2 \left(\frac{n_k t}{2} \right) \right)^2 dt dx.$$

By Parseval's equality this is

$$16 \int_{-\pi}^{\pi} \left(\sum (a_k^2 + b_k^2) \sin^4 \left(\frac{n_k t}{2} \right) \right) \frac{dt}{|t|^3}.$$

We have

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^4 \left(\frac{n_k t}{2} \right) \frac{dt}{|t|^3} &= 2 \int_0^{\pi/n_k} \sin^4 \left(\frac{n_k t}{2} \right) \frac{dt}{t^3} + 2 \int_{\pi/n_k}^{\pi} \sin^4 \left(\frac{n_k t}{2} \right) \frac{dt}{t^3} \\ &\leq 2 \int_0^{\pi/n_k} \left(\frac{n_k t}{2} \right)^4 \frac{dt}{t^3} + 2 \int_{\pi/n_k}^{\pi} \frac{dt}{t^3} \\ &= \left(\frac{n_k}{2} \right)^4 \cdot \left(\frac{\pi}{n_k} \right)^2 + \left(\frac{n_k^2}{2} - \frac{1}{\pi^2} \right) = O(n_k^2), \end{aligned}$$

and hence the integral (6) is bounded by $16D \sum_k (a_k^2 + b_k^2) n_k^2 < \infty$. This completes the proof of the theorem.

We add some corollaries.

COROLLARY 1. *Suppose f has a lacunary Fourier series. Then if f has an approximate derivative in a set of positive measure f has a derivative almost everywhere.*

PROOF. This follows from the fact (see e.g. [1, Volume II, page 263]) that if a function f has a lacunary Fourier series which converges on a set of positive measure to a function which can be extended to an absolutely continuous one, then $\sum_j (a_j^2 + b_j^2) n_j^2 < \infty$.

COROLLARY 2. *Let $0 < \alpha < 1$, $\alpha + \beta = 1$, and f have a lacunary Fourier series. Then f has an α th derivative in a set E of positive measure and hence almost everywhere if and only if for almost every x in E there is an $\eta_x = \eta > 0$ such that*

$$(7) \quad \int_{-\eta}^{\eta} \frac{R_x(t)^2}{|t|^{1+2\alpha}} dt < \infty.$$

PROOF. Let $F(x) = f_{\beta}(x)$. Assuming (7), we have $F(x)$ has a derivative in the L^2 sense and hence an approximate derivative almost

everywhere in E . By Corollary 1, $F(x)$ has an ordinary derivative almost everywhere. On the other hand, if f has an α th derivative almost everywhere in E , then $F(x)$ has an ordinary derivative almost everywhere again by Corollary 1. By Theorem 1, f satisfies the condition Λ_α almost everywhere. Hence, by Theorem A, (7) holds almost everywhere.

This corollary extends to the case $\alpha > 1$ by applying Taylor's theorem as in Theorem 1.

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