ON THE TRANSFORM OF A SINGULAR OR AN ABSOLUTELY CONTINUOUS MEASURE

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Let G be a locally compact abelian group with dual Γ. A well-known theorem of Bochner [1], generalized by Eberlein [2], is the following:

"A continuous function φ defined on Γ is the Fourier-Stieltjes transform of a finite (regular) measure on G if and only if there is a constant A such that for every trigonometric polynomial

$$p(x) = \sum c_i(-x, \gamma_i), \qquad \gamma_i \in \Gamma,$$

the relation $\|p\|_\infty \leqq 1$ implies $\left| \sum c_i \varphi(\gamma_i) \right| \leqq A$."

If we take $A = \sup \left| \sum c_i \varphi(\gamma_i) \right|$ where the sup is over all polynomials p for which $\|p\|_\infty \leqq 1$, then whatever be $\epsilon > 0$, there is a polynomial

$$p(x) = \sum c_i(-x, \gamma_i), \qquad \gamma_i \in \Gamma,$$

such that $\|p\|_\infty \leqq 1$ and $\left| \sum c_i \varphi(\gamma_i) \right| > A - \epsilon$.

A splitting of the last property gives very simple (mutually exclusive) characterizations of the transform of a singular or an absolutely continuous measure. We prove the following theorems:

THEOREM 1. *A continuous function φ defined on Γ is the Fourier-Stieltjes transform of a singular measure on G if and only if there is a constant A such that*

(i) *for every trigonometric polynomial $p(x) = \sum c_i(-x, \gamma_i)$, $\gamma_i \in \Gamma$, the relation $\|p\|_\infty \leqq 1$ implies $\left| \sum c_i \varphi(\gamma_i) \right| \leqq A$;*

(iis) *whatever be $\epsilon > 0$ and the compact set K in Γ there is a polynomial*

$$p(x) = \sum c_i(-x, \gamma_i), \qquad \gamma_i \in \Gamma,\ \gamma_i \notin K,$$

such that $\|p\|_\infty \leqq 1$ and $\left| \sum c_i \varphi(\gamma_i) \right| > A - \epsilon$.

THEOREM 2. *A continuous function φ defined on Γ is the transform of an absolutely continuous measure on G if and only if*

(i) *there is a constant A such that for every polynomial*

$$p(x) = \sum c_i(-x, \gamma_i), \qquad \gamma_i \in \Gamma,$$

the relation $\|p\|_\infty \leqq 1$ implies $\left| \sum c_i \varphi(\gamma_i) \right| \leqq A$;

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(iia) *whatever be* $\epsilon > 0$ *there is a compact set* K *in* Γ *such that for every polynomial* $p(x) = \sum c_i(-x, \gamma_i)$, $\gamma_i \in \Gamma$, $\gamma_i \notin K$, *the relation* $\|p\|_\infty \leqq 1$ *implies* $\left| \sum c_i \varphi(\gamma_i) \right| < \epsilon$.

PROOF OF THEOREM 1.

Necessity of (iis). Let $\varphi(\gamma) = \hat{\mu}_s(\gamma)$ where μ_s is singular and $\|\mu_s\| = A$. Let $\epsilon > 0$ and the compact set K in Γ be given. There is a polynomial

$$r(x) = \sum_{j=1}^{N} b_j(-x, \gamma_j'), \qquad \gamma_j' \in \Gamma,$$

such that $\|r\|_\infty \leqq 1$ and $\left| \sum b_j \hat{\mu}_s(\gamma_j') \right| = \left| \int_G r(x) d\mu_s(x) \right| > A - \epsilon$.

The set $C = \{-\gamma_1', \cdots, -\gamma_N'\} \cup -K$ being compact, there is a $k \in L^1(G)$ such that $\hat{k}(\gamma) = 1$ on C; i.e., $\hat{k}(-\gamma_j') = 1$, $j = 1, \cdots, N$, and $\hat{k}(-\gamma) = 1$ for $\gamma \in K$ and such that $\|k\|_1 < 1 + \epsilon$ (see, e.g., [3, p. 53]). Put $k'(x) = k(-x)$ and $f(x) = (k' * \mu_s)(x)$. Then $f \in L^1(G)$ and $\hat{f}(\gamma) = \hat{k}(-\gamma)\hat{\mu}_s(\gamma)$ for $\gamma \in \Gamma$. Also

$$\left| \int_G r(x) f(x) dx \right| = \left| \sum b_j \hat{f}(\gamma_j') \right| = \left| \sum b_j \hat{k}(-\gamma_j') \hat{\mu}_s(\gamma_j') \right|$$

$$= \left| \sum b_j \hat{\mu}_s(\gamma_j') \right| > A - \epsilon.$$

Hence $\|f\|_1 > A - \epsilon$.

Now consider the measure $d\mu = d\mu_s - f(x)dx$. We have $\|\mu\| = \|\mu_s\| + \|f\|_1 > 2A - \epsilon$. Hence there exists a polynomial $q(x) = \sum d_i(-x, \gamma_i)$ such that $\|q\|_\infty \leqq 1$ and $\left| \sum d_i \hat{\mu}(\gamma_i) \right| > 2A - \epsilon$. Put $p(x) = \frac{1}{2}[q(x) - (q*k)(x)]$. Then

$$p(x) = \sum c_i(-x, \gamma_i) \tag{1}$$

where $c_i = \frac{1}{2}[d_i - d_i \hat{k}(-\gamma_i)]$ so that

$$c_i = 0 \quad \text{for } \gamma_i \in K. \tag{2}$$

Also,

$$\|p\|_\infty \leqq \tfrac{1}{2} + \tfrac{1}{2}(1 + \epsilon) = 1 + \epsilon/2. \tag{3}$$

Now

$$\sum c_i \hat{\mu}_s(\gamma_i) = \tfrac{1}{2} \sum \left[d_i \hat{\mu}_s(\gamma_i) - d_i \hat{\mu}_s(\gamma_i) \hat{k}(-\gamma_i) \right]$$

$$= \tfrac{1}{2} \sum d_i [\hat{\mu}_s(\gamma_i) - \hat{f}(\gamma_i)] = \tfrac{1}{2} \sum d_i \hat{\mu}(\gamma_i).$$

Hence

$$\left| \sum c_i \hat{\mu}_s(\gamma_i) \right| > A - \epsilon/2. \tag{4}$$

Since $\epsilon > 0$ is arbitrary, relations (1) and (2), (3) and (4) prove the necessity of (iis).

Sufficiency of (i) *and* (iis). We know already by Bochner's theorem that φ is the transform of a regular finite measure μ, where $\|\mu\| = A$. We show that μ is singular. Let $d\mu = d\mu_s + g(x)dx$ be the Lebesgue decomposition of μ. $\epsilon > 0$ being given, there is an $h \in L^1(G)$, whose transform $\hat{h}$ vanishes outside some compact set K, and such that $\|g-h\|_1 < \epsilon$. Put $dv = d\mu - h(x)dx = d\mu_s + (g-h)dx$. Then $\hat{v}(\gamma)$ coincides with $\varphi(\gamma)$ outside K. Put $A_s = \|\mu_s\|$. By (iis) there is a polynomial

$$p(x) = \sum c_i(-x, \gamma_i), \qquad \gamma_i \notin K,$$

such that $\|p\|_\infty \leqq 1$ and $\left| \sum c_i(\hat{g}(\gamma_i) - \hat{h}(\gamma_i)) + \sum c_i \hat{\mu}_s(\gamma_i) \right| > A - \epsilon$. The l.h.s. is at most $A_s + \epsilon$. Therefore, $A_s + \epsilon \geqq A - \epsilon$, i.e., $A_s \geqq A - 2\epsilon$. Since $A_s \leqq A$ we conclude $A_s = A$ and therefore $\|g\|_1 = 0$. Thus μ is singular and the proof is complete.

PROOF OF THEOREM 2.

Necessity of (iia). Let φ be the transform $\hat{f}$ of some $f \in L^1(G)$ and let $\epsilon > 0$ be given. There is a $k \in L^1(G)$ with compact support K, such that $\|f - f * k\|_1 < \epsilon$. If now $p(x) = \sum c_i(-x, \gamma_i)$, $\gamma_i \in \Gamma$, $\gamma_i \notin K$ with $\|p\|_\infty \leqq 1$, then, for $p'(x) = p(-x)$,

$$\begin{aligned} \left| \sum c_i \varphi(\gamma_i) \right| &= \left| \sum c_i \hat{f}(\gamma_i) \right| = \left| \sum c_i(\hat{f}(\gamma_i) - \hat{k}(\gamma_i)\hat{f}(\gamma_i)) \right| \\ &= \left| (p' * (f - k * f))(0) \right| \leqq \|p'\|_\infty \cdot \|f - k * f\|_1 < \epsilon. \end{aligned}$$

This proves the necessity of (iia).

Sufficiency of (i) *and* (iia). We know already that φ is the transform $\hat{\mu}$ of some finite measure on G. Let $d\mu = d\mu_s + f(x)dx$ be the Lebesgue decomposition of μ. Put $A_s = \|\mu_s\|$. Let $\epsilon > 0$ be given. Let K_1 be the compact set in Γ associated to ϵ by (iia), and let K_2 be the compact set associated to ϵ and the absolutely continuous measure $f(x)dx$ by the necessity just proved. Put $K = K_1 \cup K_2$. By Theorem 1 there is a polynomial $p(x) = \sum c_i(-x, \gamma_i)$, $\gamma_i \in \Gamma$, $\gamma_i \notin K$, such that $\|p\|_\infty \leqq 1$ and $\left| \sum c_i \hat{\mu}_s(\gamma_i) \right| > A_s - \epsilon$. Also, $\left| \sum c_i \varphi(\gamma_i) \right| < \epsilon$ since $K_1 \subset K$ and $\left| \sum c_i \hat{f}(\gamma_i) \right| < \epsilon$ since $K_2 \subset K$. We conclude

$$A_s - \epsilon < \left| \sum c_i \hat{\mu}_s(\gamma_i) \right| = \left| \sum c_i(\varphi(\gamma_i) - \hat{f}(\gamma_i)) \right| < 2\epsilon,$$

i.e., $A_s < 3\epsilon$. Since ϵ is arbitrary, we conclude $A_s = 0$ so that $d\mu_s = 0$ and μ is absolutely continuous. This completes the proof of the theorem.

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