SHORTER NOTES

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A NOTE ON UNICELLULAR OPERATORS

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A bounded linear operator A on a (complex) Banach space is *unicellular* if its lattice of invariant subspaces is totally ordered under inclusion. Two examples of unicellular operators are discussed in [**1**].

THEOREM 1. *If A is an operator on a Banach space and M is an invariant subspace of A that is comparable with every other invariant subspace of A, then M is invariant under every operator that commutes with A.*

PROOF. Let $AB=BA$. Choose a complex number λ such that $|\lambda| > \|B\|$. Then $(B-\lambda)^{-1}$ and $(B-\lambda)$ have the same invariant subspaces by a result of Sarason's, [2, p. 53]. The commutativity implies that the subspace $(B-\lambda)M$ is an invariant subspace of A, and thus that either $(B-\lambda)M\subset M$ or $(B-\lambda)M\supset M$. In the first case the proof is finished. If $(B-\lambda)M\supset M$ then $M\supset(B-\lambda)^{-1}M$, and therefore M is invariant under $(B-\lambda)^{-1}$ and hence also under $(B-\lambda)$.

COROLLARY 1. *If A is unicellular and B commutes with A then every invariant subspace of A is invariant under B.*

THEOREM 2. *If A is a unicellular operator on a separable Banach space then the set of cyclic vectors of A is a residual set.*

PROOF. We must show that the set S of vectors that are not cyclic for A is a first category set. If $\{M_\alpha\}$ is the family of all proper invariant subspaces of A then clearly $S=\bigcup M_\alpha$. If the closure of S is not the whole space then S is contained in a proper subspace and hence is nowhere dense. Suppose, then, that the closure of S is the whole space. Let $\{x_i\}$ be a countable dense subset of S. For each i choose an M_{α_i} such that x_i is in M_{α_i}. It suffices to show that $S=\bigcup M_{\alpha_i}$, for then S will be exhibited as a countable union of nowhere dense sets. Consider any M_α. If M_α is not contained in $\bigcup M_{\alpha_i}$ then M_α contains M_{α_i} for all i. Thus M_α contains $\{x_i\}$ and therefore M_α is the whole space.

COROLLARY 2. *If A is a unicellular operator on a separable Banach space then every invariant subspace of A is cyclic.*

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PROOF. If M is any invariant subspace of A then $A|M$ is unicellular and hence Theorem 2 applies.

It is easily seen that in the finite-dimensional case an operator is unicellular if and only if it is cyclic and its spectrum contains only one point. An interesting but no doubt difficult question is whether or not every unicellular operator in the infinite-dimensional case must have only one point in its spectrum.

REFERENCES

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2. Donald Sarason, *The H^p spaces of an annulus*, Mem. Amer. Math. Soc. No. 56 (1965).

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