ON UNIVERSAL SPACES OF THE TYPE $C(X)$

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In [1] Banach showed that every separable Banach space can be linearly isometrically embedded in $C[0, 1]$. It is natural to ask for which compact Hausdorff space X is it true that every separable Banach space can be linearly isometrically embedded in $C(X)$. A characterization of such X 's is given in this paper when $C(X)$ is the real Banach space of all continuous real-valued functions on X . It is also shown that if $C(X)$ is not universal for all separable Banach spaces, then only a very restricted class of separable Banach spaces can be linearly isometrically embedded in $C(X)$. In particular, for each nonuniversal $C(X)$ with $\dim C(X) > 1$ there is a two dimensional space which is not embeddable in $C(X)$. A characterization of those Banach spaces that can be linearly isometrically embedded in $C(X)$ for every infinite compact Hausdorff space X is also given.

For a given Banach space B , B^π denotes the closed unit sphere of the dual B' of B in the weak* topology. The set of extreme points of B^π is denoted by EB^π and its weak* closure is denoted by cl^*EB^π . If X is a compact Hausdorff space, e denotes the evaluation map of X into $C(X)'$. It is well known that e is a homeomorphism into $C(X)'$ with the weak* topology and that $EC(X)' = e(X) \cup -e(X)$ [2, p. 86]. If $i: B \rightarrow C(X)$ is a linear isometry into, then the adjoint i' of i maps $C(X)^\pi$ onto B^π , and, in fact, $i'(EC(X)^\pi) \supseteq EB^\pi$ [3, p. 441].

The following proposition shows that not all infinite compact spaces X have the property that $C(X)$ is universal for the set of all separable Banach spaces.

1. PROPOSITION. *Let B be a Banach space. Then B is linearly, isometrically, embeddable in a $C(X)$ for X dispersed if and only if cl^*EB^π is dispersed (see [6] for results on dispersed spaces, i.e., spaces without perfect subsets).*

PROOF. If $i: B \rightarrow C(X)$ is a linear isometry, then $i'(EC(X)^\pi) \supseteq EB^\pi$. Hence, if X is dispersed, then so is $EC(X)^\pi$ and thus, so is cl^*EB^π . Conversely, if cl^*EB^π is dispersed, then B is linearly, isometrically, embeddable in $C(X)$, where $X = \text{cl}^*EB^\pi$ [2, p. 93].

In particular, since the two dimensional Euclidean space $B = l^2(2)$ has the property that EB^π is not dispersed, it is not linearly and isometrically embeddable in $C(X)$ whenever X is dispersed. It is now

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shown that this property characterizes dispersed spaces and is also equivalent to the nonuniversality of $C(X)$.

2. THEOREM. *Let X be a compact Hausdorff space. Then the following are equivalent.*

- (a) X is dispersed,
- (b) $C(X)$ contains no linear subspace B of $\dim > 1$ such that every element of norm 1 in B^π is also in EB^π (that is, B' is rotund),
- (c) $C(X)$ contains no linear subspace B of $\dim > 1$ such that the unit sphere of B is supported by only one hyperplane at each of its boundary points (that is, B is smooth),
- (d) $C(X)$ does not contain $l^2(2)$ linearly and isometrically,
- (e) $C(X)$ is not universal for the set of all separable Banach spaces.

PROOF. (a) implies (b). If $B \subseteq C(X)$ and X is dispersed, then cl^*EB^π is dispersed. Hence, if $\dim B > 1$, then EB^π is not all of the boundary points of B^π .

(b) implies (c). Suppose $B \subseteq C(X)$, $\dim B > 1$, and B is smooth. If $B_1 \subseteq B$ is finite dimensional, $\dim B_1 > 1$, then B_1 is smooth and hence B_1' is rotund [2, p. 112].

(c) implies (d). $l^2(2)$ is smooth.

(d) implies (e). $l^2(2)$ is separable.

(e) implies (a). If X is not dispersed, then there is a continuous function from X onto $[0, 1]$ (this is a simple modification of Urysohn's lemma to utilize perfectness, see [6] also). Thus $C[0, 1]$ is linearly and isometrically embeddable in $C(X)$.

A characterization of the Banach spaces that can be linearly and isometrically embedded in $C(X)$ for every infinite compact space X is now given. The space c is the Banach space of all real convergent sequences and c_0 is the space of all real sequences converging to 0.

3. THEOREM. *Let B be a Banach space. Then the following are equivalent.*

- (a) B is linearly and isometrically embeddable in $C(X)$ for all infinite compact spaces X ,
- (b) B is linearly and isometrically embeddable in c ,
- (c) $EB^\pi = Z \cup -Z$, where Z is countable, $Z \cap -Z = \emptyset$, and Z has at most one weak* limit point.

PROOF. That (a) implies (b) is clear. (b) implies (c). Let i be the linear isometry of B into c . Then $\text{cl}^*EB^\pi \subset i'(Ec^\pi)$ and $Ec^\pi = \{\delta n, \delta w, -\delta n, -\delta w\}$ where $\delta_k(j) = 0$ if $k \neq j$ and $\delta_k(k) = 1$ for $j, k = 1, \dots, w$. Also, $\delta_k \rightarrow \delta w$ in the weak* topology. Let k_1 be the smallest integer such that $i'(\delta_{k_1}) \in EB^\pi$ and let $Z_1 = i'(\delta_{k_1})$. Suppose

$k_1 < \dots < k_n$ have been chosen so that $Z_j = i'(\delta_{k_j}) \in EB^\pi$, $i'(\delta_j) \notin EB^\pi$ for $j \neq k_i$, $i = 1, \dots, n$; $j < kn$, and $Z_i \neq -Z_j$ for $i, j = 1, \dots, n$. If $\{\pm Z_j \mid 1 \leq j \leq n\} = EB^\pi$, then $Z = \{Z_1, \dots, Z_n\}$. If not, let k_{n+1} be the smallest integer such that $i'(\delta_{k_{n+1}}) \in EB^\pi$, $i'(\delta_{k_{n+1}}) \neq \pm Z_j$, $j = 1, \dots, n$, and set $Z_{n+1} = i'(\delta_{k_{n+1}})$. Thus, by induction one obtains a countable set Z such that $Z \cap -Z = \emptyset$, $Z \cup -Z = EB^\pi$. Since $\{\delta_k\}$ has only one weak* limit point, Z has a weak* limit point if and only if it is infinite, and, in such a case the limit point is $i'(\delta w)$.

(c) implies (a). It is first shown that B is linearly and isometrically embeddable in c , and then that c is linearly and isometrically embeddable in $C(X)$. If $Z = \{Z_1, \dots, Z_n\}$ then the map $i(x)(k) = \langle x, Z_k \rangle$ if $1 \leq k \leq n$, $i(x)(k) = 0$ for $k > n$ is a linear isometry of B into c (actually into c_0). If $Z = \{Z_1, Z_2, \dots\}$, then $\{Z_n\}$ converges in the weak* topology to some Z_w . The map $i(x)(k) = \langle x, Z_k \rangle$ for $1 \leq k \leq w$ is a linear isometry of B into c . Now, if X is not dispersed, then by (e) of Theorem 2, $C(X)$ is universal for all separable Banach spaces. Suppose X is an infinite dispersed compact space. It suffices to show that there is a continuous map of $N = \{k \mid 1 \leq k \leq w\}$. It is clear that X has infinitely many isolated points. Let $I = \{sn \mid n = 1, 2, \dots\}$ be a countable collection of distinct isolated points of X . Define $h: X \rightarrow N$ by $h(sn) = n$ and $h(x) = w$ for all $x \in X \setminus I$. Since a subset of N is closed if and only if it is either finite or contains w . Thus, the preimage under h of a closed set is closed.

4. COROLLARY. B is linearly and isometrically embeddable in c_0 if and only if $EB^\pi = Z \cup -Z$, where $Z \cap -Z = \emptyset$, Z is either finite or the only weak* limit point of Z is 0.

PROOF. If i is the embedding of B into c_0 , then $EB^\pi \subset i'(Ec_0^\pi)$. Now, $Ec_0^\pi = \{\delta n, -\delta n\}$ and δn converges to 0 in weak* topology. Thus, if Z is chosen as in (b) implies (c) of Theorem 3, either Z is finite or converges to 0 in the weak* topology.

The map defined in (c) implies (a) of Theorem 3 shows that if $EB^\pi = Z \cup -Z$, $Z \cap -Z = \emptyset$, Z is finite or converges to 0 in the weak* topology, then B is embeddable in c_0 .

EXAMPLE. There is a two dimensional space B such that EB^π is infinite and B is linearly and isometrically embeddable in c . Let $\theta n = (n-1/n)(\pi/2)$ and $\theta w = \pi/2$ and $\cup$ be the convex symmetric hull of $\{(\cos \theta k, \sin \theta k) \mid 1 \leq k \leq w\}$. Then $\cup$ is a closed, bounded, absorbing, symmetric, convex set and hence is the unit sphere of its Minkowski norm. If A denotes this space and $B = A'$, then B is embeddable in c by Theorem 3.

It is now shown that this is not true for c_0 . Namely, a finite dimensional space B is linearly and isometrically embeddable in c_0 if and only if EB^* is finite.

5. DEFINITION. A finite dimensional space B is said to be polyhedral if its unit sphere has only a finite number of extreme points.

For results on polyhedral sets see [4] and [5]. The following proposition is given in [2, p. 266] also.

6. PROPOSITION. *Let B be a finite dimensional Banach space. Then B is polyhedral if and only if it is linearly and isometrically embeddable in c_0 .*

PROOF. Suppose B is polyhedral. Then so is B' [5, p. 83]. Thus EB^* is finite and by Corollary 4 B is embeddable in c_0 . Conversely, if B is linearly and isometrically embeddable in c_0 , then $EB^* = Z \cup -Z$, where either Z is finite or has 0 as its only weak* limit point. Since B is finite dimensional, the latter case cannot happen. Hence Z is finite, that is, B' is polyhedral. Thus $B'' = B$ is polyhedral.

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