SPECTRAL MULTIPLICITY OF SELFADJOINT DILATIONS

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Let A be a closed symmetric operator in a Hilbert space H . If A^+ is a selfadjoint operator in a Hilbert space H^+ such that $H \subset H^+$ and $A \subset A^+$, then we shall call A^+ a *dilation* of A . In the particular case that $H = H^+$, A^+ is also called an *extension* of A . A dilation A^+ is called *minimal* if A^+ is not reduced by $H^+ \ominus H$ nor by any of its non-trivial subspaces. For further information on selfadjoint extensions and dilations of symmetric operators, we refer the reader to Achieser and Glasmann [1].

Suppose that A has deficiency indices $(1, 1)$. It is the purpose of this article to show that if μ_0 is a point of regular type for A , then there is a neighborhood of μ_0 in which every minimal selfadjoint dilation of A has spectral multiplicity not exceeding 1. More precisely, there exists an interval (γ, δ) containing μ_0 such that if A^+ is a minimal selfadjoint dilation of A with spectral function $E^+(\lambda)$, then the operator $A^+[E^+(\delta) - E^+(\gamma)]$ is unitarily equivalent to the multiplication operator in a space $L^2(\rho)$ for some nondecreasing function ρ defined on the real line. This constitutes a generalization of Theorem 5.2 of McKelvey [4], which is in turn a generalization of Theorem 14 of Coddington [2].

As a preliminary we prove two lemmas.

LEMMA 1. *A real point μ_0 is a point of regular type for a closed symmetric operator A if and only if there is a selfadjoint extension of A for which μ_0 is a regular point (i.e., a point in the resolvent set).*

PROOF. Suppose that A_0 is a selfadjoint extension of A for which μ_0 is a regular point. Since $(A_0 - \mu_0 E)^{-1}$ exists and is bounded it follows that $(A - \mu_0 E)^{-1}$ exists and is bounded. Hence, μ_0 is a point of regular type for A .

Suppose, on the other hand, that μ_0 is a point of regular type for A . By Hartman [3, §3, item (ii)], there exists a selfadjoint extension A_0 such that μ_0 is not in the point spectrum of A_0 . Therefore, $(A_0 - \mu_0 E)^{-1}$ exists. By McKelvey [4, Lemma 4.2], $(A_0 - \mu_0 E)^{-1}$ is bounded and has domain H . Hence, μ_0 is in the resolvent set of A_0 . The lemma is proved.

LEMMA 2. *Suppose that A is a closed symmetric operator with defi-*

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ciency indices $(1, 1)$. Let $g(\lambda_0)$ be an element of norm 1 in the deficiency subspace of A corresponding to the nonreal number $\bar{\lambda}_0$. Let (γ, δ) be a finite real interval in the resolvent set of a selfadjoint extension A_0 of A . Suppose that $g(\lambda) = g(\lambda_0) + (\lambda - \lambda_0)R_0(\lambda)g(\lambda_0)$ and that $\rho_0(\mu) = (E_0(\mu)g(\lambda_0), g(\lambda_0))$, where $R_0(\lambda)$ is the resolvent and $E_0(\mu)$ the spectral function of A_0 . Let $\{\gamma_n\}$ be a decreasing sequence of real numbers approaching $-\infty$, and let $\{\delta_n\}$ be an increasing sequence of real numbers approaching $+\infty$, where $\gamma_1 < \gamma < \delta < \delta_1$. Let

$$h_n = \{1 - [\rho_0(\delta_n) - \rho_0(\gamma_n)]\}^{-1} \{g(\lambda_0) - [E_0(\delta_n) - E_0(\gamma_n)]g(\lambda_0)\}.$$

Then as n approaches $+\infty$, $(g(\lambda), h_n)$ approaches 1, uniformly for $\gamma < \lambda < \delta$.

PROOF. We note first of all that since A_0 is unbounded, it follows that $\rho_0(\delta_n) - \rho_0(\gamma_n) \neq 1$, so that our definition of h_n is legitimate.

A straightforward calculation with use of the operational calculus shows that if $\gamma < \lambda < \delta$,

$$\begin{aligned} (g(\lambda), g(\lambda_0) - [E_0(\delta_n) - E_0(\gamma_n)]g(\lambda_0)) &= 1 - [\rho_0(\delta_n) - \rho_0(\gamma_n)] \\ &+ (\lambda - \lambda_0) \int_{-\infty}^{\gamma_n} (\mu - \lambda)^{-1} d(E_0(\mu)g(\lambda_0), g(\lambda_0)) \\ &+ (\lambda - \lambda_0) \int_{\delta_n}^{\infty} (\mu - \lambda)^{-1} d(E_0(\mu)g(\lambda_0), g(\lambda_0)). \end{aligned}$$

Hence, for each $\epsilon > 0$ there exists an N such that if $n \geq N$ and $\gamma < \lambda < \delta$,

$$\begin{aligned} |(g(\lambda), h_n) - 1| &\leq \{1 - [\rho_0(\delta_n) - \rho_0(\gamma_n)]\}^{-1} \\ &\cdot \{\epsilon(E_0(\gamma_n)g(\lambda_0), g(\lambda_0)) + \epsilon[1 - (E_0(\delta_n)g(\lambda_0), g(\lambda_0))]\} = \epsilon. \end{aligned}$$

This proves the lemma.

THEOREM 1. Suppose that A is a closed symmetric operator with deficiency indices $(1, 1)$. Let A_0 be a selfadjoint extension of A . Suppose that $[\gamma_1, \delta_1]$ is a real interval in the resolvent set of A_0 . Then, every minimal selfadjoint dilation of A has spectral multiplicity no more than 1 in $[\gamma_1, \delta_1]$.

PROOF. Let A^+ be any minimal selfadjoint dilation of A . Let $R(\lambda)$ be the generalized resolvent and $E(\lambda)$ the spectral function of A defined by A^+ . As is known, for all f, h in H and for any real numbers α, β , $\alpha < \beta$,

$$\begin{aligned} &(\tfrac{1}{2}\{E(\beta) + E(\beta + 0)\} - \tfrac{1}{2}\{E(\alpha) + E(\alpha + 0)\})f, h) \\ (1) \quad &= (2\pi i)^{-1} \lim_{\tau \rightarrow +0} \int_{\alpha}^{\beta} ([R(\sigma + i\tau) - R(\sigma - i\tau)]f, h) d\sigma. \end{aligned}$$

(See Štraus [5], for example.)

According to Krein (see Achieser and Glasmann [1, Appendix I, §4]), for any nonreal λ ,

$$(2) \quad R(\lambda) = R_0(\lambda) - [\theta(\lambda) + Q_1(\lambda)]^{-1}(\cdot, g(\bar{\lambda}))g(\lambda).$$

Here $R_0(\lambda)$ is the resolvent of A_0 ; $g(\lambda) = g(\lambda_0) + (\lambda - \lambda_0)R_0(\lambda)g(\lambda_0)$, Where $I\lambda_0 > 0$ and $g(\lambda_0)$ is an element of norm 1 in the deficiency subspace of A corresponding to $\bar{\lambda}_0$; $Q_1(\lambda) = iI\lambda_0 + (\lambda - \lambda_0)(g(\lambda_0), g(\bar{\lambda}))$, and $Q_1(\lambda)$ has the properties that $Q_1(\bar{\lambda}) = [Q_1(\lambda)]^-$ and that $IQ_1(\lambda) > 0$ for $I\lambda > 0$; $\theta(\lambda)$ is an analytic function in the upper and lower halfplanes with the properties that $\theta(\bar{\lambda}) = [\theta(\lambda)]^-$ and that $I\theta(\lambda) \geq 0$ for $I\lambda > 0$. All of these are proved in the section just cited from Achieser and Glasmann, except the facts that $Q_1(\bar{\lambda}) = [Q_1(\lambda)]^-$, that $IQ_1(\lambda) > 0$ for $I\lambda > 0$, and that $\theta(\bar{\lambda}) = [\theta(\lambda)]^-$. Here $[-]$ stands for complex conjugate.

One may check that $Q_1(\bar{\lambda}) = [Q_1(\lambda)]^-$ as follows: Substituting the expression for $g(\lambda)$ into $Q_1(\bar{\lambda})$, we obtain

$$\begin{aligned} Q_1(\bar{\lambda}) &= iI\lambda_0 + (\bar{\lambda} - \lambda_0)(g(\lambda_0), g(\lambda_0)) + (\bar{\lambda} - \lambda_0)(\bar{\lambda} - \bar{\lambda}_0)(g(\lambda_0), R_0(\lambda)g(\lambda_0)) \\ &= -iI\lambda_0 + (\bar{\lambda} - \bar{\lambda}_0)(g(\lambda_0), g(\lambda_0)) + (\bar{\lambda} - \lambda_0)(\bar{\lambda} - \bar{\lambda}_0)(R_0(\bar{\lambda})g(\lambda_0), g(\lambda_0)) \\ &= -iI\lambda_0 + (\bar{\lambda} - \bar{\lambda}_0)(g(\bar{\lambda}), g(\lambda_0)) = [Q_1(\lambda)]^-. \end{aligned}$$

One may prove that $IQ_1(\lambda) > 0$ for $I\lambda > 0$ as follows:

$$\begin{aligned} g(\bar{\lambda}) &= g(\lambda_0) + (\bar{\lambda} - \lambda_0)R_0(\bar{\lambda})g(\lambda_0) = (A_0 - \lambda_0 E)(A_0 - \bar{\lambda} E)^{-1}g(\lambda_0) \\ &= \int_{-\infty}^{\infty} (\mu - \lambda_0)(\mu - \bar{\lambda})^{-1} dE_0(\mu)g(\lambda_0), \end{aligned}$$

where $E_0(\mu)$ is the spectral function of A_0 . Hence,

$$\begin{aligned} Q_1(\lambda) &= iI\lambda_0 + (\lambda - \lambda_0) \int_{-\infty}^{\infty} (\mu - \bar{\lambda}_0)(\mu - \lambda)^{-1} d\|E_0(\mu)g(\lambda_0)\|^2 \\ &= iI\lambda_0 + \int_{-\infty}^{\infty} [(\bar{\lambda}_0 - \mu) + |\mu - \lambda_0|^2(\mu - \lambda)^{-1}] d\|E_0(\mu)g(\lambda_0)\|^2. \end{aligned}$$

The last equation implies that if $\lambda = x + iy$, then,

$$IQ_1(\lambda) = y \int_{-\infty}^{\infty} |\mu - \lambda_0|^2 |\mu - \lambda|^{-2} d\|E_0(\mu)g(\lambda_0)\|^2 > 0.$$

That $\theta(\bar{\lambda}) = [\theta(\lambda)]^-$ is proved by using the expression

$$\theta(\lambda) = -\|g(\bar{\lambda})\|^2 \|g(\lambda)\|^2 \{([R(\lambda) - R_0(\lambda)]g(\bar{\lambda}), g(\lambda))\}^{-1} - Q_1(\lambda),$$

which follows from equation (2).

If we substitute (2) into (1), we obtain

$$\begin{aligned}
 & \left(\left[\frac{1}{2} \{ E(\beta) + E(\beta + 0) \} - \frac{1}{2} \{ E(\alpha) + E(\alpha + 0) \} \right] f, h \right) \\
 &= - (2\pi i)^{-1} \lim_{\tau \rightarrow +0} \int_{\alpha}^{\beta} \{ [\theta(\sigma + i\tau) + Q_1(\sigma + i\tau)]^{-1} (f, g(\sigma - i\tau)) \\
 (3) \quad & \cdot (g(\sigma + i\tau), h) - [\theta(\sigma - i\tau) + Q_1(\sigma - i\tau)]^{-1} \\
 & \cdot (f, g(\sigma + i\tau)) (g(\sigma - i\tau), h) \} d\sigma.
 \end{aligned}$$

If we put $\Psi(\lambda) = (f, g(\bar{\lambda}))(g(\lambda), h)$ and $\Phi(\lambda) = -[\theta(\lambda) + Q_1(\lambda)]^{-1}$ and suppose that $[\alpha, \beta] \subset [\gamma_1, \delta_1]$, we may use the properties of $\theta(\lambda)$, $Q_1(\lambda)$ and $g(\lambda)$ to check that $\Psi(\lambda)$ and $\Phi(\lambda)$ satisfy the hypotheses of Štraus [5, Lemma 5]. Hence,

$$\begin{aligned}
 & \left(\left[\frac{1}{2} \{ E(\beta) + E(\beta + 0) \} - \frac{1}{2} \{ E(\alpha) + E(\alpha + 0) \} \right] f, h \right) \\
 (4) \quad &= \int_{\alpha}^{\beta} (f, g(\sigma))(g(\sigma), h) d\rho(\sigma)
 \end{aligned}$$

where

$$(5) \quad \rho(\sigma) = - (1/\pi) \lim_{\tau \rightarrow +0} \int_0^{\sigma} I[\theta(\eta + i\tau) + Q_1(\eta + i\tau)]^{-1} d\eta.$$

It follows from (4) that

$$(6) \quad ([E^+(\beta) - E^+(\alpha)]f, h) = \int_{[\alpha, \beta]} (f, g(\sigma))(g(\sigma), h) d\rho(\sigma),$$

where $E^+(\mu)$ is the spectral function of A^+ , and the integral is in the sense of Lebesgue.

If $\Delta = [\alpha, \beta]$, we define $E^+(\Delta)$ by the equation $E^+(\Delta) = E^+(\beta) - E^+(\alpha)$. Let $\Delta_0 = [\gamma_1, \delta_1]$. As in McKelvey [3, Lemma 3.1], the closed linear hull of $Z(\Delta_0) = \{ E^+(\Delta)h : \Delta \subset \Delta_0, h \in H \}$ is $E^+(\Delta_0)H^+$.

We define the operator V on $Z(\Delta_0)$ into $L^2_{\rho}[\gamma_1, \delta_1]$ as follows: $VE^+(\Delta)h = \chi_{[\alpha, \beta]}(\lambda)(h, g(\lambda))$, where $\Delta = [\alpha, \beta] \subset [\gamma_1, \delta_1]$, and $\chi_{[\alpha, \beta]}(\lambda)$ is the characteristic function of $[\alpha, \beta]$. We then extend V linearly to the linear hull of $Z(\Delta_0)$. Using Lemma 2, it is not difficult to check that if $E^+(\Delta)h = E^+(\Delta')h'$, then $VE^+(\Delta)h = VE^+(\Delta')h'$, and similarly for elements in the linear hull of $Z(\Delta_0)$. Further, from (6) it follows that V is an isometry. Extending V continuously to $E^+(\Delta_0)H^+$, we obtain an isometry from $E^+(\Delta_0)H^+$ into $L^2_{\rho}[\gamma_1, \delta_1]$.

V is, in fact, onto $L^2_{\rho}[\gamma_1, \delta_1]$. For, suppose that $k(\lambda) \in L^2_{\rho}[\gamma_1, \delta_1]$ and that $k(\lambda)$ is perpendicular to $\chi_{[\alpha, \beta]}(\lambda)(h, g(\lambda))$ for every $[\alpha, \beta] \subset [\gamma_1, \delta_1]$ and for all $h \in H$. Therefore, $\int_{[\alpha, \beta]} (h_n, g(\lambda))k(\lambda)d\rho(\lambda) = 0$ for all

$[\alpha, \beta) \subset [\alpha_1, \beta_1)$, where $\{h_n\}$ is the sequence of Lemma 2. Taking the limit, it follows from Lemma 2 that $\int_{[\alpha, \beta)} k(\lambda) d\rho(\lambda) = 0$ for all $[\alpha, \beta) \subset [\gamma_1, \delta_1)$. Hence, $k(\lambda)$ is perpendicular to all step functions in $L^2_\rho[\gamma_1, \delta_1)$. The density of the step functions implies that $k(\lambda) = 0$ almost everywhere. Whence, V is onto.

It is not difficult to check that V carries the spectral function of $A^+E^+(\Delta_0)$ into the spectral function of the multiplication operator in $L^2_\rho[\gamma_1, \delta_1)$. It follows that the spectrum of $A^+E^+(\Delta_0)$ has multiplicity 0 or 1, depending on whether ρ is constant or nonconstant in $[\gamma_1, \delta_1)$. This proves the theorem.

By using Lemma 1 and Theorem 1 we obtain the following:

THEOREM 2. *Suppose that A is a closed symmetric operator with deficiency indices $(1, 1)$. Let μ_0 be a point of regular type for A . Then there is a neighborhood of μ_0 in which every minimal selfadjoint dilation of A has spectral multiplicity not exceeding 1.*

REMARK. One might ask whether there might always be a neighborhood of a real point μ_0 of regular type in which the spectral multiplicity of every minimal selfadjoint dilation is zero. This is not true, because there exists a selfadjoint extension for which μ_0 is an eigenvalue. (See Achieser and Glasmann [1, §83].) Further, there will in general be actual dilations (rather than extensions) for which the spectral multiplicity is different from zero in every neighborhood of μ_0 . This can be seen from the fact that there are many symmetric operators with deficiency index $(1, 1)$ for which every point in the complex plane is of regular type. (For example, operators defined by formally selfadjoint first order ordinary differential operators with zero boundary conditions on a finite interval, operators defined by certain Sturm-Liouville operators on a semi-axis, and operators defined by certain Jacobi matrices.) Such a symmetric operator A has an infinite number of selfadjoint dilations (actual dilations). Since a selfadjoint operator cannot have spectral multiplicity zero at every point, for each such selfadjoint dilation A^+ of A there will be at least one point μ_0 (which is of necessity a point of regular type for A) in every neighborhood of which the spectral multiplicity of A^+ is greater than zero. Such points will not even in general be eigenvalues, because in [2] and [4] it is shown that many of these dilations will be unitarily equivalent to the multiplication operator in the Hilbert space of Lebesgue square integrable functions on the real line.

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