NEUMANN'S LEMMA¹

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1. Introduction. The purpose of this note is to fill an apparent gap in existing proofs of a result of C. Neumann. The result, known now as Neumann's Lemma, appears in his book [5, Chapter V] and is mentioned in Kellogg [4], Goursat [2], Kantorovich and Krylov [3], Warschawski [9], Gaier [1], and elsewhere. Historically, Neumann's Lemma was used to solve the Dirichlet problem of potential theory for bounded convex regions after Weierstrass and others had pointed out that the solution to the problem via the Dirichlet principle was not always valid. More recently, Neumann's Lemma has been used to show convergence and to give explicit error estimates of methods for constructing the harmonic function of the Dirichlet problem (e.g., [3]) and also the Riemann mapping function (e.g., [1], [9]) for bounded convex regions.

2. Neumann's Lemma. We restrict ourselves to a bounded convex region D in the complex plane, but we make no additional smoothness assumptions concerning the boundary of D . Additional smoothness assumptions are made in [1], [2], [3], but the proofs still contain gaps which will be described in §3.

Just the boundedness and convexity of D insures that ∂D is a rectifiable Jordan curve. Moreover, $C = \partial D$ is a curve of *bounded rotation*; that is, one-sided tangents exist at every point of C , and the angles which they make with a fixed direction are of bounded variation with respect to arc length. So except for a countable set, a tangent angle τ is defined and continuous with respect to arc length, and its discontinuities are jumps which can be assumed to be at most π in modulus. In fact, since C cannot have cusps, $\Theta \equiv \max_s |\tau(s+0) - \tau(s-0)| < \pi$.

Represent C as a function of arc length by the equation $z = \zeta(s)$, $0 \leq s \leq L$, in such a way that $s = 0$ is a point of continuity of τ . If μ is a continuous function on C and a branch of $\arg(\zeta - z)$ is determined for each $z \notin C$, then

$$w(z) = \pi^{-1} \int_0^L \mu(\zeta(t)) d \arg(\zeta(t) - z), \quad z \notin C,$$

is the potential of a double layer on C with density μ .

For boundary value problems one is interested in $w_i(s)$ and $w_e(s)$,

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the interior and exterior boundary values of $w(z)$, respectively. Following Radon [6] we may represent

$$w_i = (I + T)\mu, \quad -w_e = (I - T)\mu$$

where I is the identity, the operator T is defined by

$$(Tf)(s) = \int_0^L f(t) d\psi(s, t),$$

and the function

$$\begin{aligned} \psi(s, t) &= \pi^{-1} \arg(\zeta(t) - \zeta(s)) && \text{for } 0 \leq t < s \leq L, \\ &= \pi^{-1} \arg(\zeta(t) - \zeta(s)) + 1 && \text{for } 0 \leq s < t \leq L, \\ &= \psi(s, s + 0) && \text{for } 0 \leq s = t < L, \\ &= \psi(s, s - 0) && \text{for } s = t = L. \end{aligned}$$

Branches of the argument function are chosen so that for $s \neq t$ the function ψ is continuous and $\psi(s, t) = \psi(t, s)$. Since C is convex, $\psi(s, \cdot)$ is nondecreasing and $\int_0^L d\psi(s, t) = 1$ for all s . For points $\zeta(s) \in C$ at which the tangent angle τ is continuous,

$$(Tf)(s) = \int_0^L f(t) K(s, t) dt$$

where $K(s, t)$ is the classical Neumann-Poincaré kernel of two-dimensional potential theory (cf. [8]).

It is easily verified that T is a bounded linear operator on $\mathcal{C}[0, L]$, the space of continuous functions on $[0, L]$ with the sup norm. Using the notation $\text{osc } f = \sup f - \inf f$ on $[0, L]$, we now state Neumann's result.

THEOREM 1 (NEUMANN'S LEMMA). *Let C be any convex Jordan curve which is not the boundary of a quadrilateral or a triangle. Then there exists a constant k , $0 \leq k < 1$, depending only on C , such that for all $f \in \mathcal{C}[0, L]$*

$$(1) \quad \text{osc } Tf \leq k \text{ osc } f.$$

REMARK 1. If $R \leq \infty$ is the supremum of the radii of all circles which intersect C in at least three points, then there exists a $k \leq 1 - L/(2\pi R)$. This estimate was derived by Neumann [5, p. 176] with the exception of an improvement in the last term by a factor of two which carries over from proofs under smoothness assumptions found in [1], [3].

REMARK 2. The peculiar exception of quadrilaterals and triangles

is necessary. For these figures no $k < 1$ exists. To see this, let $\zeta(s_1)$ and $\zeta(s_2)$ be two vertices (nonadjacent for a quadrilateral) and define a function f as one at $\zeta(s_1)$ and on the segments adjacent to $\zeta(s_2)$ and as zero elsewhere, including at $\zeta(s_2)$. Then f can be approximated in an obvious way by a sequence of continuous functions f_n for which $\text{osc } f_n = 1$, $\lim_{n \rightarrow \infty} (Tf_n)(s_1) = 1$, and $\lim_{n \rightarrow \infty} (Tf_n)(s_2) = 0$. Consequently, $\lim_{n \rightarrow \infty} \text{osc } Tf_n = 1$ so that $\text{osc } Tf \leq k \text{ osc } f_n$ holds uniformly only if $k \geq 1$.

3. Proof and remarks. We shall first prove a lemma and then point out the apparent gaps in existing proofs of Theorem 1.

LEMMA 1. *If C is any convex Jordan curve which is not the boundary of a quadrilateral or a triangle, then for all $f \in \mathcal{C}[0, L]$, $\text{osc } Tf < \text{osc } f$.*

PROOF. Let the minimum of $f \in \mathcal{C}[0, L]$ be m and choose $s_1, s_2 \in [0, L]$ so that $\text{osc } Tf = (Tf)(s_1) - (Tf)(s_2)$. Then since $\int_0^L d\psi(s, t) \equiv 1$,

$$\begin{aligned} \text{osc } Tf &= \int_0^L f(t) d[\psi(s_1, t) - \psi(s_2, t)] \\ &= \int_0^L [f(t) - m] d[\psi(s_1, t) - \psi(s_2, t)] \leq (\text{osc } f) V \end{aligned}$$

where V is the total upper variation of $\psi(s_1, \cdot) - \psi(s_2, \cdot)$ on $[0, L]$. Let E and E' denote the Hahn decomposition of $[0, L]$ relative to $\psi(s_1, \cdot) - \psi(s_2, \cdot)$ so that

$$V = \int_E d[\psi(s_1, t) - \psi(s_2, t)] = 1 - \int_{E'} d\psi(s_1, t) - \int_E d\psi(s_2, t).$$

Since $\psi(s, \cdot)$ is nondecreasing, $V \leq 1$ and equality occurs only if both

$$\int_{E'} d\psi(s_1, t) = 0, \quad \int_E d\psi(s_2, t) = 0$$

and so only if

$$(2) \quad \int_{E' - \{s_1\}} d \arg(\zeta(t) - \zeta(s_1)) = 0, \quad \int_{E - \{s_2\}} d \arg(\zeta(t) - \zeta(s_2)) = 0.$$

Since $\arg(\zeta(t) - \zeta(s_1))$ and $\arg(\zeta(t) - \zeta(s_2))$ are absolutely continuous (with appropriate one-sided limits at s_1 and s_2 , respectively), it follows from (2) that there exists a dense family of intervals on each of which either $\arg(\zeta(t) - \zeta(s_1))$ or $\arg(\zeta(t) - \zeta(s_2))$ is constant. Since C is convex, this can occur only if $\zeta(E')$ and $\zeta(E)$ lie on straight line segments

issuing from $\zeta(s_1)$ and $\zeta(s_2)$, respectively. The latter is excluded by hypothesis so that $V < 1$, and the lemma is proved.

REMARKS. The crucial step in the preceding proof was that

$$(3) \quad V = \int_E d[\psi(s_1, t) - \psi(s_2, t)] < 1.$$

However, this inequality was derived without regard to the choice of the points s_1, s_2 and the set E . Neumann [5, p. 173] concluded that the integral $V(s_1, s_2)$ in (3) therefore had a "maximum value" $k < 1$ from which Theorem 1 would follow. Goursat [2, p. 201] noted that this conclusion did not follow but felt that it should be so "for curves which one usually considers." Kantorovich and Krylov [3] remarked that, for smooth curves, $V(s_1, s_2)$ is a continuous function of s_1 and s_2 for every fixed (measurable) set E and so has a maximum. Since it is the total upper variation which is needed, the set E must also be considered as a function of s_1 and s_2 in order to employ a continuity argument. Hence a gap still exists. Gaier [1] has a proof without any gaps, but it is valid only for convex curves whose curvature varies continuously *and never vanishes*.

PROOF OF THEOREM 1. To remove the difficulties involving the set E mentioned above, note that the total upper and lower variations of $\psi(s_1, \cdot) - \psi(s_2, \cdot)$ are equal since $\int_0^L d[\psi(s_1, t) - \psi(s_2, t)] = 0$. Therefore

$$(4) \quad V(s_1, s_2) = \frac{1}{2} \int_0^L |d[\psi(s_1, t) - \psi(s_2, t)]| < 1$$

for all $s_1, s_2 \in [0, L]$.

From the fact (see [6] or [7, p. 225]) that

$$(5) \quad \limsup_{s_1 - s_2 \rightarrow 0} \int_0^L |d[\psi(s_1, t) - \psi(s_2, t)]| = \Theta/\pi,$$

it follows easily that $V(s_1, s_2)$ is a continuous function of s_1 and s_2 if and only if $\Theta = 0$. Thus if C has a continuously turning tangent, then $V(s_1, s_2)$ is continuous and has a maximum value $k < 1$, so that the proof of Theorem 1 is easily completed under this smoothness assumption. However, we complete the proof without any smoothness assumptions.

It is sufficient to show that $V(s_1, s_2) \leq k < 1$ uniformly for all $s_1, s_2 \in [0, L]$. If this were not true, then there would exist convergent sequences $s_1^n \rightarrow \sigma_1$ and $s_2^n \rightarrow \sigma_2$ such that for every n

$$(6) \quad 1 - 1/n < V(s_1^n, s_2^n) < 1.$$

Necessarily $\sigma_1 \neq \sigma_2$, for otherwise by (4), (5), and (6),

$$\begin{aligned}
1 &= \lim_{n \rightarrow \infty} V(s_1^n, s_2^n) \leq \limsup_{n \rightarrow \infty} V(s_1^n, \sigma_1) + \limsup_{n \rightarrow \infty} V(s_2^n, \sigma_2) \\
&\leq (\Theta/2\pi) + (\Theta/2\pi) = (\Theta/\pi) < 1.
\end{aligned}$$

Since $\psi(s, t)$ is continuous for $s \neq t$, we estimate for $0 < \epsilon < \frac{1}{2} |\sigma_1 - \sigma_2|$ and $i = 1, 2$ that

$$\begin{aligned}
J_i &= \limsup_{n \rightarrow \infty} \int_{\sigma_i - \epsilon}^{\sigma_i + \epsilon} |d[\psi(s_1^n, t) - \psi(s_2^n, t)]| \\
&\leq \limsup_{n \rightarrow \infty} \left\{ \int_{\sigma_i - \epsilon}^{\sigma_i + \epsilon} d\psi(s_1^n, t) + \int_{\sigma_i - \epsilon}^{\sigma_i + \epsilon} d\psi(s_2^n, t) \right\} \\
&= \limsup_{n \rightarrow \infty} \{ \psi(s_1^n, \sigma_i + \epsilon) - \psi(s_1^n, \sigma_i - \epsilon) + \psi(s_2^n, \sigma_i + \epsilon) \\
&\quad - \psi(s_2^n, \sigma_i - \epsilon) \} \\
&= \psi(\sigma_i, \sigma_i + 0) - \psi(\sigma_i, \sigma_i - 0) + o(\epsilon).
\end{aligned}$$

Let H_ϵ be those points of $[0, L]$ separated by at least ϵ from both σ_1 and σ_2 . (Appropriate modifications should be introduced if σ_1 or σ_2 is an endpoint of $[0, L]$.) Then

$$\begin{aligned}
1 &= \lim_{n \rightarrow \infty} V(s_1^n, s_2^n) \\
&\leq \frac{1}{2} \left\{ J_1 + J_2 + \limsup_{n \rightarrow \infty} \int_{H_\epsilon} |d[\psi(s_1^n, t) - \psi(s_2^n, t)]| \right\}.
\end{aligned}$$

On H_ϵ the functions $\psi(s_1^n, \cdot) - \psi(s_2^n, \cdot)$ are absolutely continuous, converge to $\psi(\sigma_1, \cdot) - \psi(\sigma_2, \cdot)$, and their one-sided derivatives converge boundedly (cf. [6, p. 1147]). Therefore

$$\begin{aligned}
1 &\leq \frac{1}{2} \left\{ \sum_{i=1}^2 [\psi(\sigma_i, \sigma_i + 0) - \psi(\sigma_i, \sigma_i - 0)] + o(\epsilon) \right. \\
(7) \quad &\quad \left. + \int_{H_\epsilon} |d[\psi(\sigma_1, t) - \psi(\sigma_2, t)]| \right\}.
\end{aligned}$$

Letting $\epsilon \rightarrow 0$ in (7), one has $1 \leq V(\sigma_1, \sigma_2)$. This contradicts (4) and completes the proof.

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