HILL'S EQUATIONS WITH HALF-PERIODIC COEFFICIENTS

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We shall concern ourselves with Hill's differential equation

$$(1) \quad y'' + [\lambda + qf(t)]y = 0.$$

$f(t)$ is to be a real function of period π , such that $\int_0^\pi |f(t)| dt < \infty$, and λ, q are parameters. Let y_1 and y_2 denote two solutions of (1) characterized by their initial values

$$y_1(0) = y_2'(0) = 1, \quad y_1'(0) = y_2(0) = 0.$$

The function

$$\Delta(\lambda, q) = y_1(\pi) + y_2'(\pi)$$

is known as the discriminant of Hill's equation and depends on the parameters λ and q . It is well known that for all λ and q for which

$$(2) \quad \Delta(\lambda, q) = 2$$

(1) has solutions of period π . For all λ and q for which

$$(3) \quad \Delta(\lambda, q) = -2$$

(1) has solutions of period 2π . For proofs of these facts and other pertinent background material see Hochstadt [1], Magnus and Winkler [2].

It is also known that the functions $\lambda(q)$ satisfying (2) or (3) are analytic functions of q for small q , see Hochstadt [3]. If we stipulate that $f(t)$ be an even function of t and $\lambda(q)$ satisfies (2), then in general periodic solutions of (1) will be either even or odd. If even, they can be characterized by the boundary conditions

$$(4) \quad y_1'(0) = y_1'(\pi/2) = 0$$

and if odd, by

$$(5) \quad y_2(0) = y_2(\pi/2) = 0.$$

The corresponding solutions of (2) will be denoted by $\{\lambda_i(q)\}$.

If $\lambda(q)$ satisfies (3) the periodic solutions of (1) will again be either even or odd. The even solutions are characterized by

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$$(6) \quad y_1'(0) = y_1(\pi/2) = 0$$

and the odd ones by

$$(7) \quad y_2(0) = y_2'(\pi/2) = 0.$$

The corresponding solutions of (3) will be denoted by $\{\lambda_i'(q)\}$.

We are now in a position to state and prove the following theorem.

THEOREM. *Consider Hill's equation (1) where $f(t)$ is real integrable and such that*

$$f(t + \pi) = f(t), \quad f(-t) = f(t)$$

and also satisfies

$$f(t + \pi/2) = -f(t).$$

Let $\lambda_i(q)$ be a solution of (2), and $\lambda'_{2i-1}(q)$, $\lambda'_{2i}(q)$ solutions of (3) such that $\lambda_i(0) = (2i)^2$, $\lambda'_{2i-1}(0) = \lambda'_{2i}(0) = (2i+1)^2$. Then

$$\lambda_i(-q) = \lambda_i(q), \quad \lambda'_{2i-1}(-q) = \lambda'_{2i}(q).$$

In other words eigenvalues corresponding to the boundary conditions (4) and (5) are even functions of q and eigenvalues corresponding to (6) and (7) are interchanged upon replacement of q by $-q$.

REMARK. For the particular case where $f(t) = \cos 2t$, (1) reduces to the Mathieu equation. The results of this theorem are well known for this special case. The proof in Meixner and Schaeffe [4], however, does not apply to the general case discussed here.

PROOF. Suppose $\lambda_i(q)$ satisfies (2) and a solution corresponding to (4) exists. It can be expressed by

$$y(t) = \sum_{n=0}^{\infty} a_n \cos 2nt.$$

Upon replacement of t by $t + \pi/2$ in (1) we obtain

$$(8) \quad y'' + [\lambda_i(q) - qf(t)]y = 0$$

and it must be satisfied by

$$y(t + \pi/2) = \sum_{n=0}^{\infty} a_n (-1)^n \cos 2nt.$$

Then $\lambda_i(q)$ is an eigenvalue of (1) corresponding to the boundary conditions (4), as well as of (8) under the same boundary conditions. Thus we see that $\lambda_i(-q) = \lambda_i(q)$. Since it is an analytic function of q we have

$$\lambda_i(q) = \sum_{n=0}^{\infty} \mu_{n,i} q^{2n}.$$

The situation regarding eigenvalues with odd eigenfunctions follows in a similar manner.

We now consider the eigenvalues $\lambda'_{2i-1}(q)$ and $\lambda'_{2i}(q)$. Suppose that corresponding to $\lambda'_{2i-1}(q)$, (1) has the even solution

$$y_1 = \sum_{n=0}^{\infty} b_n \cos(2n+1)t$$

and corresponding to $\lambda'_{2i}(q)$, (1) has the odd solution

$$y_2 = \sum_{n=0}^{\infty} c_n \sin(2n+1)t.$$

These solutions satisfy (6) and (7) respectively. If, as before, we replace t by $t + \pi/2$ we see that

$$y_1(t + \pi/2) = - \sum_{n=0}^{\infty} b_n (-1)^n \sin(2n+1)t$$

satisfies

$$(9) \quad y'' + [\lambda'_{2i-1}(q) - qf(t)]y = 0$$

and

$$y_2(t + \pi/2) = \sum_{n=0}^{\infty} c_n (-1)^n \cos(2n+1)t$$

satisfies

$$(10) \quad y'' + [\lambda'_{2i}(q) - qf(t)]y = 0.$$

Thus we see that if $\lambda'_{2i-1}(q)$ corresponds to (1) having even solutions (9) will have odd solutions and correspondingly for $\lambda'_{2i}(q)$. Then if

$$\lambda'_{2i-1}(q) = \sum_{n=0}^{\infty} \nu_{n,i} q^n, \quad \lambda'_{2i}(q) = \sum_{n=0}^{\infty} \nu_{n,i} (-1)^n q^n,$$

thus completing the proof of the theorem.

It is possible to answer another question with this theorem. An open question regarding Hill's equation concerns the inverse problem. This problem poses the question as to whether a knowledge of the discriminant allows one to reconstruct the function $f(t)$. It is known that the answer is negative, but it was conjectured that if $f(t)$ is an even function then an affirmative answer might be possible. The

preceding theorem shows us that if $f(t)$ is even and half-periodic, then the two equations

$$y'' + [\lambda \pm f(t)]y = 0$$

have the same discriminant. Hence, even in this restricted class of Hill's equations the discriminant does not determine $f(t)$.

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