AN ALGEBRAIC CHARACTERIZATION OF CERTAIN ORTHONORMAL SYSTEMS[1]

J. J. PRICE

1. Introduction. The system $\Phi = \{\phi_n\}_0^\infty$ of Haar functions is an orthonormal set in $L^2[0, 1]$ with the following algebraic properties:

(a) $\phi_0(x) \equiv$ constant.

(b) If $n > k$, $\phi_n \phi_k = c_{nk} \phi_n$ where c_{nk} is a constant, possibly zero.

(c) ϕ_n^2 is a linear combination of $\phi_0, \phi_1, \cdots, \phi_{n-1}$.

In this paper, we prove that the above properties characterize orthonormal sets that are essentially certain subsets of the Haar system. In fact, (b) and (c) may be replaced by a more general assumption.

(d) For each $n \geq 0$, the linear space spanned by $\{\phi_0, \phi_1, \cdots, \phi_n\}$ is a ring. These results follow from Theorem 1, in which weaker assumptions are made and a wider class of orthonormal sets is characterized.

Generalizations and related results are given in §4.

2. Definitions and notation. All functions will be real-valued and belong to $L^2(X, \mu)$ where (X, μ) is a totally finite measure space.

(ϕ, ψ) will denote the usual inner product in $L^2(X, \mu)$.

For a given set of functions Φ, $\Lambda(\Phi)$ will denote the set of all finite linear combinations of elements of Φ.

For a given function ϕ, the set where ϕ is nonzero will be denoted by $\sigma(\phi)$ and will be called the support of ϕ.

If $P = \{X_i\}$ is a partition of X into a finite number of subsets, or parts, of positive measure, $\nu(P)$ will be the number of parts. If P and P' are partitions of X, we write $P' < P$ if P' refines P but $P' \neq P$. I will denote the identity partition. A function constant on each part of a partition P will be called a step function (P).

Certain orthonormal systems will be called Haar sets (for details, see [1], [2]). Consider a sequence $\{P_i\}_0^\infty$ of partitions of the following type.

(i) $P_0 = I$.

(ii) $P_0 > P_1 > P_2 > \cdots$.

(iii) P_{i+1} is formed by splitting one part of P_i into two new parts. Observe that (i) and (iii) imply $\nu(P_i) = i+1$. There is an orthonormal

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set $\{\phi_i\}_0^\infty$, unique except for choice of signs, such that ϕ_0 is a constant, and for $i \geq 1$, ϕ_i is constant on each of the new parts of P_i and vanishes otherwise. Such an orthonormal system we call a Haar set. When X is the unit interval and at each stage the two new parts have equal measures, the corresponding Haar set is a subsystem of the ordinary Haar functions. (Not every subsystem of the Haar functions, however, is a Haar set.)

The notion of a Haar set can be generalized in the following way. Let $\{P_i\}_0^\infty$ be a sequence of partitions such that

(iv) $P_0 > P_1 > P_2 > \cdots$, $\nu(P_i) = n_i + 1$ where $\{n_i\}_0^\infty$ is an increasing sequence of integers,

(v) P_{i+1} is formed by splitting one part of P_i into $n_{i+1} - n_i + 1$ new parts.

There are orthonormal sets $\{\phi_n\}_0^\infty$ with the property that for each i, the functions ϕ_n, $n_i < n \leq n_{i+1}$, are constant on the new parts of P_{i+1} and vanish otherwise ($n_{-1} = -1$). These will be called generalized Haar sets.

We remark that it is meaningful to speak also of finite sets of this type.

3. **The main theorem.** We now characterize generalized Haar sets by certain algebraic properties less restrictive than (a), (b), and (c) of the introduction.

THEOREM 1. *Let* $\Phi = \{\phi_n\}_0^\infty$ *be an orthonormal set in* $L^2(X, \mu)$. *Suppose*

(1) $\phi_0(x)$ *does not vanish.*

(2) *If* $n > k$, $\phi_n \phi_k \in \Lambda(\phi_0, \phi_1, \cdots, \phi_n)$.

(3) *For each* n, $\phi_n^2 \in \Lambda(\Phi)$. *Then, after a suitable reordering,* Φ *is a generalized Haar set.*

PROOF. By hypothesis (2),

$$(4) \qquad \phi_n \phi_k = \sum_{\nu=0}^n a_{nk}^{(\nu)} \phi_\nu, \quad \text{when } n > k.$$

Let $j < n$, $j \neq k$.

$$a_{nk}^{(j)} = (\phi_n \phi_k, \phi_j) = (\phi_n, \phi_k \phi_j) = \left(\phi_n, \sum_{\nu=0}^l a_{kj}^{(\nu)} \phi_\nu\right) = \sum_{\nu=0}^l a_{kj}^{(\nu)} (\phi_n, \phi_\nu)$$

where $l = \max\{k, j\} < n$. By orthogonality, $a_{nk}^{(j)} = 0$, and so the expansion (4) reduces to

$$(5) \qquad \phi_n \phi_k = a_{nk}^{(n)} \phi_n + a_{nk}^{(k)} \phi_k.$$

We introduce an equivalence relation into the set Φ as follows:

$$\phi_n \approx \phi_k \equiv \phi_n\phi_k = a\phi_n + b\phi_k, \quad ab \neq 0 \quad (n \neq k); \quad \phi_n \approx \phi_n.$$

The transitivity of this relation is easily checked. If $\phi_n \sim \phi_k$ and $\phi_m \sim \phi_n$, and if we assume no two of n, m, k are equal, as we may, then

$$(6) \qquad \phi_n\phi_k = a\phi_n + b\phi_k, \qquad \phi_m\phi_n = c\phi_m + d\phi_n$$

where all coefficients are nonzero. Multiplying the first of the above relations by ϕ_m, the second by ϕ_k and equating yields

$$(b - c)\phi_m\phi_k = d\phi_n\phi_k - a\phi_m\phi_n.$$

Substituting relations (6) in the above equation yields

$$(b - c)\phi_m\phi_k = bd\phi_k - ac\phi_m.$$

Both coefficients on the right-hand side above are nonzero. Therefore $b - c$ must be nonzero, and so $\phi_m \sim \phi_k$.

Let $[\phi_n]$ denote the equivalence class of ϕ_n. We assert that each class $[\phi_n]$ is finite. If $\phi_n \sim \phi_k$, and $n \neq k$, then

$$(\overset{2}{\phi_n}, \phi_k) = (\phi_n, \phi_n\phi_k) = (\phi_n, \overset{(n)}{a_{nk}}\phi_n + \overset{(k)}{a_{nk}}\phi_k) = \overset{(n)}{a_{nk}} \neq 0.$$

But hypothesis (3) guarantees that (ϕ_n^2, ϕ_k) is nonzero for only a finite number of functions ϕ_k. Therefore $[\phi_n]$ is finite.

If ϕ_n is not equivalent to ϕ_k, there are three possibilities. Either $\phi_n\phi_k = a_{nk}^{(n)}\phi_n$, $\phi_n\phi_k = a_{nk}^{(k)}\phi_k$, or $\phi_n\phi_k = 0$. We now make the following assumptions about the enumeration of our system Φ.

(7) If $\phi_n\phi_k = a_{nk}^{(n)}\phi_n$, where $a_{nk}^{(n)} \neq 0$, then $n > k$.

(8) The members of each equivalence class are consecutively indexed. It will be shown later that Φ can be reordered so that (7) and (8) are satisfied.

Because of (8), there is an increasing sequence of integers $\{n_i\}_{-1}^{\infty}$, $n_{-1} = -1$, such that each equivalence class is a set of the form $\{\phi_j : n_i < j \leq n_{i+1}\}$.

Let $\Lambda_n = \Lambda(\phi_0, \phi_1, \cdots, \phi_n)$. We claim Λ_{n_i} is an algebra over the reals. It suffices to show $\phi_j\phi_k \in \Lambda_{n_i}$ whenever $\phi_j, \phi_k \in \Lambda_{n_i}$. For $j \neq k$, this is so by hypothesis (2). $\phi_k^2 \in \Lambda_{n_i}$ if ϕ_k^2 is orthogonal to ϕ_r whenever $r > n_i$. Now Λ_{n_i} contains $[\phi_k]$. Therefore, ϕ_k is not equivalent to ϕ_r, and so, using (7),

$$(\overset{2}{\phi_k}, \phi_r) = (\phi_k, \phi_r\phi_k) = \overset{(r)}{a_{rk}}(\phi_k, \phi_r) = 0.$$

Hence $\phi_k^2 \in \Lambda_{n_i}$.

We conclude that $\Lambda(\Phi)$ is an algebra of the form $\bigcup_{i=0}^{\infty} \Lambda_{n_i}$ where Λ_{n_i} is a subalgebra with orthonormal basis $\{\phi_0, \phi_1, \cdots, \phi_{n_i}\}$.

Consider a fixed subalgebra Λ_{n_i}. If $\phi_k \in \Lambda_{n_i}$, then since the dimension of Λ_{n_i} is n_i+1, the functions $\phi_k, \phi_k^2, \cdots, \phi_k^{n_i+2}$ are linearly dependent. Therefore ϕ_k can assume only a finite number of values. Consequently, Λ_{n_i} induces a finite number of separation classes in X, and may be regarded as an algebra of continuous real functions that separate points of a finite (compact) space X' and do not all vanish at any given point. By an elementary form of the Stone-Weierstrass Theorem, Λ_{n_i} is the set of all real functions on X', and card X' $= \dim \Lambda_{n_i} = n_i+1$. In other words, Λ_{n_i} is the space of all step functions (P_i) where P_i is the partition of X into separation classes induced by Λ_{n_i} and $\nu(P_i) = n_i+1$.

Since $\Lambda_{n_i} \subset \Lambda_{n_{i+1}}$, the separation classes of $\Lambda_{n_{i+1}}$ are subsets of those of Λ_{n_i}. Hence $P_{i+1} < P_i$. In order to establish that Φ is a generalized Haar set, it will suffice to show that all elements of $\Lambda_{n_{i+1}} - \Lambda_{n_i}$ vanish on each part of P_i except one, for it will then follow that only one part of P_i splits when P_{i+1} is formed. The set $\Lambda_{n_{i+1}} - \Lambda_{n_i} = \{\phi_k : n_i < k \leq n_{i+1}\}$ is an equivalence class. Since equivalent functions have identical supports, as is evident from (5), all we need prove is that the support of any of these functions ϕ_k is contained in one part of P_i.

Such a function ϕ_k is not equivalent to any member of Λ_{n_i}. Therefore, if $\phi_j \in \Lambda_{n_i}$, $\phi_k\phi_j = a_{kj}^{(k)}\phi_k$ (where $a_{kj}^{(k)}$ may be zero). By linearity, for each $\chi \in \Lambda_{n_i}$, $\phi_k\chi = c\phi_k$ where c is a constant depending on χ. Hence χ is constant on $\sigma(\phi_k)$. Since Λ_{n_i} is the space of all step functions (P_i), we may choose for χ the characteristic function of any part of P_i, say X_{i0}, that intersects $\sigma(\phi_k)$. Since χ is constant on $\sigma(\phi_k)$ and $\chi(x) = 1$ for some $x \in \sigma(\phi_k)$, $\chi(x) \equiv 1$ on $\sigma(\phi_k)$. Hence $\sigma(\phi_k) \subset X_{i0}$. Therefore Φ is a generalized Haar set.

It remains to be shown that there is a reordering of Φ such that the rearranged set $\Psi = \{\psi_n\}_0^\infty$ satisfies (7) and (8).

Equivalent elements of Φ have identical supports. If ϕ_n is not equivalent to ϕ_k, then either

$$\sigma(\phi_n) \subset \sigma(\phi_k), \quad \sigma(\phi_k) \subset \sigma(\phi_n), \quad \text{or } \sigma(\phi_n) \cap \sigma(\phi_k) = \varnothing.$$

These relations correspond to the cases

$$\phi_n\phi_k = \overset{(n)}{a_{nk}}\phi_n, \qquad \phi_n\phi_k = \overset{(k)}{a_{nk}}\phi_k, \qquad \phi_n\phi_k = 0.$$

Introduce a partial ordering into the set $[\Phi]$ of equivalence classes as follows. Define $[\phi_n] \prec [\phi_k]$ if $\sigma(\phi_n) \subset \sigma(\phi_k)$. Otherwise let $[\phi_n]$ and $[\phi_k]$ be not comparable. This is obviously a well-defined partial ordering.

Suppose $[\phi_n] \prec [\phi_k]$. Then

$$(\overset{2}{\phi_n}, \phi_k) = (\phi_n, \phi_n\phi_k) = (\phi_n, \overset{(n)}{a_{nk}}\phi_n) = \overset{(n)}{a_{nk}} \neq 0.$$

We have already pointed out that (ϕ_n^2, ϕ_k) is nonzero for only a finite number of functions ϕ_k. Hence $[\phi_n] \prec [\phi_k]$ for only a finite number of classes $[\phi_k]$. It follows that every ordered chain in $[\Phi]$ has a maximal element and the same is true if a finite number of elements are deleted from $[\Phi]$. Therefore, by Zorn's Lemma, these partially ordered sets have maximal elements.

Enumerate $[\Phi]$ as follows. As first element choose $[\phi_0]$, the unique maximal element of $[\Phi]$. Next, from the remaining equivalence classes choose the maximal element that contains ϕ_k with minimal k. Continuing in this way, we obtain an ordering in $[\Phi]$.

Using the ordering in $[\Phi]$ in an obvious manner, we can obtain an ordering $\Psi = \{\psi_n\}_0^\infty$ of Φ so that $\psi_0 = \phi_0$ and the members of each equivalence class are assigned consecutive indices. Such a rearrangement has the desired properties for if $\psi_n\psi_k = c_{nk}^{(n)}\psi_n$ where $c_{nk}^{(n)} \neq 0$, then $[\psi_n] \prec [\psi_k]$ which implies $n > k$.

This completes the proof of Theorem 1.

4. Related results. Orthonormal sets with properties (a) and (d) of the introduction satisfy the hypotheses of Theorem 1. Hence, they are generalized Haar sets with $n_i = i$. Therefore, we obtain the following corollary.

THEOREM 2. *Haar sets are characterized among orthonormal systems by properties* (a) *and* (d).

The techniques used in the proof of Theorem 1 can be applied to obtain the following result.

THEOREM 3. *Let* $\Phi = \{\phi_n\}_0^\infty$ *be an orthonormal set in* $L^2(X, \mu)$. *Suppose there is an increasing sequence* $\{n_i\}_0^\infty$ *of integers such that* Λ_{n_i} *is a ring,* $i \geq 0$. *Then there exists a decreasing sequence* $\{P_i\}_0^\infty$ *of partitions of* X *such that for each* $i \geq 0$, $\nu(P_i) = n_i + 1$ *and* Λ_{n_i} *is the set of all step functions* (P_i).

The converse of Theorem 3 is obviously true.

There is a connection between Theorem 3 and a previous result of the author. Let $\Phi = \{\phi_n\}_0^\infty$ be an orthonormal set in $L^2(X, \mu)$, $\phi_0(x) \equiv$ constant. Associated with Φ are the Dirichlet kernels

$$D_n(x, y) = \sum_{\nu=0}^{n-1} \phi_\nu(x)\phi_\nu(y).$$

In [3] it is shown that the conclusion of Theorem 3 above is necessary and sufficient for the kernels $D_{n_i+1}(x, y)$ to be nonnegative, $i \geq 0$. Therefore, we have the following theorem.

THEOREM 4. *Let $\{\phi_n\}_0^\infty$ be an orthonormal set in $L^2(X, \mu)$, $\phi_0(x)$ $\equiv$ constant. There exists a sequence $\{D_{n_i}(x, y)\}_0^\infty$ of nonnegative kernels if and only if Λ_{n_i-1} is a ring, $i \geqq 0$.*

We conclude by studying the effect of hypothesis (1) in Theorem 1.

THEOREM 5. *Suppose all assumptions of Theorem 1 hold except (1). Then there is a countable family $\{X_j\}$ of pairwise disjoint subsets of X such that the support of each function ϕ_n is contained in one of the sets X_j. Let*

$$\Phi_j = \{\phi \in \Phi: \sigma(\phi) \subset X_j\}.$$

Each Φ_j can be reordered so that the rearranged set Ψ_j is a generalized Haar set.

PROOF. Reorder Φ as in Theorem 1. Then, if $n > k$, either $\psi_n \sim \psi_k$, $\psi_n \prec \psi_k$, or ψ_n and ψ_k are not comparable. In terms of supports, either $\sigma(\psi_n) = \sigma(\psi_k)$, $\sigma(\psi_n) \subset \sigma(\psi_k)$ or, $\sigma(\psi_n)$ and $\sigma(\psi_k)$ are disjoint.

Define $X_0 = \sigma(\psi_0)$. Let ψ_{k_1} be the first element of Ψ, if any exists, whose support is disjoint from X_0 and define $X_1 = \sigma(\psi_{k_1})$. Let ψ_{k_2} be the first element whose support is disjoint from $X_0 \cup X_1$ and define $X_2 = \sigma(\psi_{k_2})$. Continuing in this way, we obtain a finite or denumerably infinite family $\{X_j\}$ of pairwise disjoint subsets of X. Every element of Ψ belongs to one of the sets

$$\Psi_j = \{\psi \in \Psi: \sigma(\psi) \subset X_j\}.$$

After an obvious renumbering, the system Ψ_j satisfies, relative to X_j, conditions (2) and (3) of Theorem 1. Condition (1) is satisfied also since $X_j = \sigma(\psi_{kj})$. Therefore Ψ_j is a generalized Haar set and the proof is complete.

We remark that some of the sets Ψ_j may be finite. But Theorem 1 obviously applies to finite orthonormal sets as well as to infinite ones.

Theorem 5 can be stated algebraically. For each j, Ψ_j contains the characteristic function of X_j. Therefore the ring $\Lambda(\Psi)$ contains a set $\{e_j\}$ of mutually orthogonal idempotents and splits into the direct sum of ideals $e_j \Lambda(\Psi) = \Lambda(\Psi_j)$.

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