A GENERALIZED LAX-MILGRAM THEOREM

EDWARD M. LANDESMAN

1. Introduction. In [2], the author introduced the Hilbert spaces H_A and $H_{A'}$ which were associated with an operator A and its $*$ -reciprocal $A' \equiv A^{*-1}$. Some general theorems relating these spaces to each other were formulated and proven.

These Hilbert spaces proved to be those which are of prime importance when studying partial differential equations, in particular linear elliptic partial differential equations. The author has shown in [2] that when A is the operator G_k , that is, the operator which maps a function into itself and its first k -derivatives, then H_{G_k} and $H_{G_k'}$ are the spaces which form the "natural" setting for the study of existence and regularity theory.

In this paper the author generalizes in a natural way the Lax-Milgram Theorem [3] by proving the theorem for the more generalized case when H_A and $H_{A'}$ are the underlying spaces. The method of proof follows that employed in [3].

2. Preliminaries. Let H and H^1 be any two real Hilbert spaces. Suppose A is an operator (closed and dense linear transformation) from H into H^1 . For simplicity, we take the null space N_A of the operator A to be zero.

Suppose that the domain D_A of A is completed with respect to the range norm $\|Ax\|$. Designate this by

$$(2.1) \quad \|x\|_A = \|Ax\|$$

for all x in the domain D_A of A . We call the Hilbert space so obtained H_A .

Consider now the closure of the range of A , $\bar{R}_A$. This is a Hilbert space with norm being the natural norm in H^1 .

Clearly, the operator A is a norm preserving map of D_A onto R_A where (2.1) is taken as the norm for D_A .

We extend A linearly to a unitary operator on H_A onto $\bar{R}_A$. We call this extension $\tilde{A}$.

Similarly, we complete the domain $D_{A'}$ of $A' \equiv A^{*-1}$, with respect to the norm

$$(2.2) \quad \|x\|_{A'} = \|A'x\|$$

for all x in the domain $D_{A'}$ of A' . We call this Hilbert space $H_{A'}$.

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As was shown in [1], the completion of the domain $D_{A'}$ with respect to the norm (2.2) is equivalent to completing D_A with respect to the norm

$$(2.3) \quad \|x\|_{A'} = \sup_{y \in D_A} \frac{(x, y)}{\|Ay\|}$$

where (x, y) denotes the ordinary inner product in H .

We call $\tilde{A}'$ the unitary operator which maps $H_{A'}$ onto $\bar{R}_{A'}$. As an immediate result, we have:

LEMMA 2.1. *The closure of the range of A is equal to the closure of the range of A' , i.e., $\bar{R}_A = \bar{R}_{A'}$.*

The result follows from the following equalities

$$\begin{aligned} (\bar{R}_A)^\perp &= R_A^\perp = N_{A^{-1}} = N_{A^*} \\ &= R_{A'}^\perp = (\bar{R}_{A'})^\perp. \end{aligned}$$

Since $\bar{R}_A$ and $\bar{R}_{A'}$ are closed, we have

$$\bar{R}_A = [(\bar{R}_A)^\perp]^\perp = [(\bar{R}_{A'})^\perp]^\perp = \bar{R}_{A'}.$$

As a simple consequence of Lemma 2.1, we see that the operator $\tilde{A}'$ maps $H_{A'}$ onto $\bar{R}_A$.

We remark that by the way we have constructed H_A and $H_{A'}$, we may always choose an x in H_A such that:

$$\tilde{A}x = \tilde{A}'z \quad \text{for a given } z \text{ in } H_{A'}.$$

This enables us to define an "inner product" between the elements of H_A and the elements of $H_{A'}$. We do so as follows:

If x is in H_A and y is in $H_{A'}$, define their inner product as

$$(2.4) \quad \langle x, y \rangle = (\tilde{A}x, \tilde{A}'y).$$

We note that if x is in D_A and y is in $D_{A'}$, then $\langle x, y \rangle = (x, y)$, the ordinary inner product in H . Henceforth, let $\| \cdot \|_{H_A}$ and $\| \cdot \|_{H_{A'}}$ denote the norms in H_A and $H_{A'}$ respectively. We immediately obtain the following theorems which have been proven in [2].

THEOREM 2.1 (GENERALIZED SCHWARZ INEQUALITY). *If x is in H_A , y is in $H_{A'}$, then*

$$(2.5) \quad |\langle x, y \rangle| \leq \|x\|_{H_A} \|y\|_{H_{A'}}.$$

THEOREM 2.2 (REPRESENTATION THEOREM). *Every bounded linear functional $L(x)$ over H_A can be represented as*

$$(2.6) \quad L(x) = \langle x, y \rangle$$

where y is in $H_{A'}$.

3. Main theorem. Let H and H^1 be real Hilbert spaces. Let A be an operator from H into H^1 and define the Hilbert spaces H_A and $H_{A'}$ as in § 2. Suppose $B(x, y)$ is a bilinear functional on $H_A \times H_{A'}$, i.e., x is in H_A and y is in $H_{A'}$.

Theorem 3.1 (GENERALIZED LAX-MILGRAM THEOREM). *Suppose that $B(x, y)$ is bounded in the following sense*

$$(3.1) \quad |B(x, y)| \leq K \|x\|_{H_A} \|y\|_{H_{A'}}$$

for some positive constant K , and for all x in H_A and for all y in $H_{A'}$.

Furthermore, suppose that there is an $M > 0$ such that

$$(3.2) \quad B(x, y) \geq M \|y\|_{H_{A'}}^2$$

for all x and y such that $\bar{A}x = \bar{A}'y$.

Then given any bounded linear functional $L(x)$ over H_A , there exists a unique y in $H_{A'}$ such that

$$(3.3) \quad L(x) = B(x, y) \quad \text{for all } x \text{ in } H_A.$$

PROOF. Assuming the existence of y , we first prove that y is unique.

If y is not unique, then there is a y_1 in $H_{A'}$ such that:

$$(3.4) \quad B(x, y) = B(x, y_1), \text{ i.e., } B(x, y - y_1) = 0 \quad \text{for all } x \text{ in } H_A.$$

Since the above is true for all x in H_A , by the way we have constructed H_A and $H_{A'}$, we may choose an x in H_A , say x_0 , such that

$$(3.5) \quad \bar{A}x_0 = \bar{A}'(y - y_1).$$

From condition (3.2), and (3.4), we get

$$(3.6) \quad M \|y - y_1\|_{H_{A'}}^2 \leq B(x_0, y - y_1) = 0 \quad \text{i.e., } y = y_1.$$

In order to prove the existence portion of the theorem, we employ a sequence of lemmas similar to what Lax and Milgram do in [3].

Define a subset S of $H_{A'}$ as follows:

S is the set of all y in $H_{A'}$ for which there is a z in H_A such that:

$$\langle x, y \rangle = B(x, z) \quad \text{for all } x \text{ in } H_A.$$

LEMMA 3.1. *The set S is a subspace of $H_{A'}$.*

Let y_1 and y_2 be in $H_{A'}$. Then $\langle x, y_1 \rangle = B(x, z_1)$ for some z_1 in H_A , and $\langle x, y_2 \rangle = B(x, z_2)$ for some z_2 in H_A .

These two equations yield $\langle x, y_1 - y_2 \rangle = B(x, z_1 - z_2)$. This shows that $y_1 - y_2$ is in $H_{A'}$.

If y is in $H_{A'}$ and r is any real number, then from

$$\langle x, y \rangle = B(x, z) \quad \text{for some } z \text{ in } H_A,$$

it follows that $\langle x, ry \rangle = B(x, rz)$ i.e., ry is in $H_{A'}$.

LEMMA 3.2. *The element z is bounded with respect to the element y .*

By the way S is defined, if y is in S , there is a z in H_A such that:

$$\langle x, y \rangle = B(x, z) \quad \text{for all } x \text{ in } H_A.$$

Choose x such that:

$$(3.7) \quad \tilde{A}x = \tilde{A}'z.$$

Then by (3.2),

$$(3.8) \quad \langle x, y \rangle = B(x, z) \geq M \|z\|_{H_A}^2.$$

Applying the Generalized Schwarz inequality (Theorem 2.1) to the left side of (3.8), we obtain:

$$(3.9) \quad \|x\|_{H_A} \|y\|_{H_A'} \geq B(x, z) \geq M \|z\|_{H_A}^2.$$

But by (3.7), $\|x\|_{H_A} = \|\tilde{A}x\| = \|\tilde{A}'z\| = \|z\|_{H_A'}$, therefore (3.9) becomes

$$(3.10) \quad \|z\|_{H_A'} \|y\|_{H_A'} \geq B(x, z) \geq M \|z\|_{H_A}^2.$$

Dividing the left and right side of (3.10) by $M \|z\|_{H_A'}$, we obtain the result:

$$\|z\|_{H_A'} \leq (1/M) \|y\|_{H_A'}.$$

LEMMA 3.3. *The set S is closed.*

Let $\{y_n\}$ be a sequence in S , $\|y_n - y\|_{H_A'} \rightarrow 0$. We must show that y is in S . Since the $\{y_n\}$ are in S , there exist $\{z_n\}$ in H_A such that:

$$\langle x, y_n \rangle = B(x, z_n) \quad \text{for all } x \text{ in } H_A.$$

By Lemma 3.2,

$$\|z_n - z_m\|_{H_A'} \leq (1/M) \|y_n - y_m\|_{H_A'}.$$

Since $\|y_n - y_m\|_{H_A'} \rightarrow 0$, we see that $\{z_n\}$ is a Cauchy sequence in H_A , and hence converges to some z in H_A .

Since by (3.1), $B(x, y)$ is continuous in both arguments, we have: $\langle x, y \rangle = \langle x, \lim_n y_n \rangle = B(x, \lim_n z_n) = B(x, z)$ for all x in H_A , i.e., y is in S and S is closed.

We are now in a position to prove our theorem. It remains to show that S is all of $H_{A'}$. This will show that all "inner products" of $H_A \times H_{A'}$ can be represented as an inner product with respect to B . By the Representation Theorem (Theorem 2.2), since every bounded linear functional $L(x)$ over H_A can be represented as an "inner product" over $H_A \times H_{A'}$, we will be through.

Suppose that S is not all of $H_{A'}$. Since S is a closed subspace of $H_{A'}$, write

$$H_{A'} = S \oplus S^\perp$$

where $S^\perp$ is nonempty and orthogonal to all of S . Let $y_0 \neq 0$ be in $S^\perp$.

Consider $B(x, y_0)$ which is a bounded linear functional of x since y_0 is fixed. By the Representation Theorem (Theorem 2.2), there exists a y in $H_{A'}$ such that:

$$(3.11) \quad B(x, y_0) = \langle x, y \rangle$$

for all x in H_A , i.e., y is in S .

Choose x such that

$$\tilde{A}x = \tilde{A}'y_0.$$

Then by (3.2), and (3.11), we have

$$(3.12) \quad M\|y_0\|_{H_{A'}}^2 \leq B(x, y_0) = \langle x, y \rangle.$$

By our choice of x , the last term of the above inequality becomes:

$$\langle x, y \rangle = (\tilde{A}x, \tilde{A}'y) = (\tilde{A}'y_0, \tilde{A}'y) = (y_0, y)_{H_{A'}} = 0,$$

since y is in S and y_0 is in $S^\perp$.

We have therefore from (3.12) that

$$M\|y_0\|_{H_{A'}}^2 \leq 0$$

i.e., $y_0 = 0$, and the proof is complete.

COROLLARY (LAX-MILGRAM THEOREM). *Let H be a real Hilbert space and $B(x, y)$ a bilinear functional which is bounded, i.e., $|B(x, y)| \leq K\|x\| \|y\|$ and positive definite, i.e., there is a constant $h > 0$ such that $B(x, x) \geq h\|x\|^2$ for all x in H . Then given any bounded linear functional $L(x)$ on H , there is a unique y in H such that $L(x) = B(x, y)$.*

In Theorem 3.1, choose $H = H^1$. Take $A = A' = I$, the identity operator. Then $H_A = H_{A'} = H$, and the corollary is an immediate special case of Theorem 3.1.

REMARK. If A is the operator G_k , which maps a function into itself and its first k derivatives, see [2], then $A' = G_k$, and the spaces H_{G_k}

and $H_{G_k'}$ are the spaces which are often used in elliptic partial differential equations. In fact H_{G_k} is the space H_k , which is the completion of the C^∞ functions under the k -Dirichlet norm, while $H_{G_k'}$ is the so called H_{-k} space which comes into play when using the negative-norms defined by P. Lax, see [4]. The author relates these concepts in [2].

With the above choice of $A = G_k$, our theorem says that any bounded linear functional over H_k is representable as a bilinear functional over $H_k \times H_{-k}$.

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UNIVERSITY OF CALIFORNIA, SANTA CRUZ