ON NORMS OF IDEMPOTENT MEASURES. II

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Let G be a locally compact abelian group and Γ its dual. The Fourier transform $\hat{\mu}$ of a measure μ on G is defined by $\hat{\mu}(\gamma) = \int_G (x, \gamma) d\mu(x)$. If $\hat{\mu}(\gamma)$ is equal to 0 or 1 for all γ in Γ , we call μ idempotent and denote by $S(\mu)$ the set of those γ for which $\hat{\mu}(\gamma) = 1$. Throughout this paper, μ will be any nonzero idempotent measure on G .

Suppose now that G is compact and that μ has the form $d\mu(x) = \{1 + (-x, \gamma)\} dm(x)$, where m denotes the Haar measure of G and γ is an element in Γ different from 0. If γ has infinite order, then

$$(1) \quad \|\mu\| = \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\theta}| d\theta = \frac{4}{\pi} \sim 1.273 \dots$$

If γ has finite order q , then

$$(2) \quad \|\mu\| = \frac{1}{q} \sum_{j=1}^q |1 + e^{2\pi i j/q}| = \begin{cases} 2/(q \sin(\pi/2q)) & (q: \text{odd}) \\ 2/(q \tan(\pi/2q)) & (q: \text{even}). \end{cases}$$

For odd q , $\|\mu\|$ decreases to $4/\pi$, as $q \rightarrow \infty$; for even q , $\|\mu\|$ increases to $4/\pi$, as $q \rightarrow \infty$. The largest value is obtained when $q=3$ and is equal to $4/3$; the smallest value larger than 1 is obtained when $q=4$ and is equal to $(2^{1/2}+1)/2 \sim 1.207 \dots$ (See Rudin [3, p. 73].)

The author has previously shown that the norm $\|\mu\|$ of a μ is not smaller than $(2^{1/2}+1)/2$ if $\|\mu\| > 1$ (to appear in the Proceedings of the American Mathematical Society). The question arises of what can be said if $\|\mu\| > (2^{1/2}+1)/2$. Furthermore, is it true that every μ with $\|\mu\| \leq 4/3$ has norm equal to $4/\pi$ or equal to one of the values in the righthand side of (2)? We shall show below that this is the case if $\|\mu\| < ((17)^{1/2}+1)/4 \sim 1.280 \dots$.

THEOREM 1. *If $1 < \|\mu\| < ((17)^{1/2}+1)/4$, then μ has the form*

$$(3) \quad d\mu(x) = \{(-x, \gamma_1) + (-x, \gamma_2)\} dm(x),$$

where m denotes the Haar measure of some compact subgroup H of G , and γ_1 and γ_2 are distinct characters of H .

This theorem is equivalent to the following one:

THEOREM 2. *Suppose that G is compact, that m is the Haar measure of G , and that μ has the form*

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$$(4) \quad d\mu(x) = \sum_{\gamma \in S} (-x, \gamma) d\mu(x)$$

for some finite subset S in Γ . Then we have $\|\mu\| \geq ((17)^{1/2} + 1)/4$ if S satisfies the following conditions:

- (a) S contains at least three elements.
- (b) S is not a union of cosets of any nontrivial subgroup of Γ .

The equivalence of Theorem 1 and Theorem 2 may be proved as follows. To show Theorem 1, we may clearly assume that G is itself the support group of μ , namely, the smallest closed subgroup of G on which μ is concentrated. We note that every μ with $\|\mu\| < 2$ has the form (4), where m is the Haar measure of some compact subgroup of G . This fact is an easy corollary of the theorem of Cohen [2]; see also [1]. Thus the assumptions of Theorem 1, combined with the above remark, imply that G is compact and that μ has the form (4). (Of course, as Rudin showed in [4], the support group of every μ is compact.) It follows that $S = S(\mu)$ contains at least two elements since $\|\mu\| > 1$. This S satisfies (b), for otherwise G would not be the support group of μ . Hence, if Theorem 2 holds, S contains precisely two elements since $\|\mu\| < ((17)^{1/2} + 1)/4$. Thus Theorem 2 implies Theorem 1. The converse is obvious since (b) assures that G is itself the support group of μ .

So we shall only prove Theorem 2. We begin with a lemma.

LEMMA. For a μ , put $S = S(\mu)$. Suppose that there exist μ and v in S and w in Γ such that $u + w$ belongs to S but neither $v + w$ nor $v - w$ belongs to S . Then we have $\|\mu\| \geq ((17)^{1/2} + 1)/4$.

PROOF. For $t > 0$, put

$$\begin{aligned} f(x) &= t(x, u)\{1 + (x, w)\} + (x, v)\{1 - \operatorname{Re}(x, w)\} \\ &= t\{(x, u) + (x, u + w)\} + (x, v) \\ &\quad - \frac{1}{2}\{(x, v + w) + (x, v - w)\}. \end{aligned}$$

The second expression for f shows $\int_G f d\mu = 2t + 1$. Setting $(x, w) = e^{2i\alpha}$, the first expression for f shows

$$\begin{aligned} |f(x)| &\leq t|1 + e^{2i\alpha}| + |1 - \cos 2\alpha| \\ &= 2t|\cos \alpha| + 2(1 - \cos^2 \alpha) \\ &= (t^2 + 4)/2 - 2(|\cos \alpha| - t/2)^2 \leq (t^2 + 4)/2. \end{aligned}$$

Hence $\|\mu\| \geq 2(2t + 1)/(t^2 + 4)$. The largest value is obtained when $t = ((17)^{1/2} - 1)/2$ and is equal to $((17)^{1/2} + 1)/4$.

We now prove Theorem 2. Since the translate of S by any element of Γ leaves the norm of μ invariant, we shall assume that S contains 0.

First suppose that S consists of precisely three elements 0 , γ_1 and γ_2 . If neither $\gamma_1 + \gamma_2$ nor $\gamma_1 - \gamma_2$ is in S , put $u = 0$, $v = \gamma_1$, $w = \gamma_2$; if $\gamma_1 + \gamma_2$ is in S , put $u = \gamma_2$, $v = 0$, $w = 2\gamma_1$; if $\gamma_1 - \gamma_2$ is in S , put $u = 0$, $v = \gamma_2$, $w = \gamma_1$. In any cases so chosen u , v , and w satisfy the assumptions of the lemma and so we have the desired conclusion.

If S contains more than three elements, the proof proceeds by induction on the number of the elements of S .

(I) Suppose that S contains a coset of the cyclic subgroup $\Gamma(\gamma_0)$ generated by a nonzero element γ_0 in Γ . In this case, we may assume, without loss of generality, that S contains $\Gamma(\gamma_0)$ itself and that the order of γ_0 is a prime.

Since S is not a union of cosets of $\Gamma(\gamma_0)$ by (b), there exists γ_1 in S such that the coset $\gamma_1 + \Gamma(\gamma_0)$ is not entirely contained in S . If S_1 is the intersection of S and $\gamma_1 + \Gamma(\gamma_0)$, it follows that S_1 is properly contained in S and satisfies (b), since the order of γ_0 is a prime. There are the following two possibilities, (i) and (ii):

(i) Suppose that S_1 contains at most two elements. If S_1 contains neither $\gamma_1 + \gamma_0$ nor $\gamma_1 - \gamma_0$, we can appeal to the lemma. In case either $\gamma_1 + \gamma_0$ or $\gamma_1 - \gamma_0$ is in S_1 we may clearly assume that $\gamma_1 + \gamma_0$ is in S_1 since γ_1 is in S_1 . Then γ_0 has order larger than 2 since S_1 contains two elements γ_1 and $\gamma_1 + \gamma_0$ and since S_1 is not a coset in Γ . It follows that none of $\gamma_1 - \gamma_0$ and $\gamma_1 + 2\gamma_0$ belongs to S_1 . If the order of γ_0 is larger than 3, S_1 contains neither $\gamma_1 + 2\gamma_0$ nor $\gamma_1 - 2\gamma_0$, and so the lemma applies if we set $u = 0$, $v = \gamma_1$, $w = 2\gamma_0$. If the order of γ_0 is equal to 3, let m_0 be the Haar measure of the annihilator of the cyclic group $\Gamma(\gamma_0)$ and let μ_0 be the convolution of m_0 and μ_1 , where $d\mu_1(x) = (x, \gamma_1)d\mu(x)$. $S(\mu_0)$ coincides with $S_1 - \gamma_1$ and so consists of two elements 0 and γ_0 , which implies $d\mu_0(x) = \{1 + (-x, \gamma_0)\}dm(x)$. We thus conclude $\|\mu\| \geq \|\mu_0\| = 4/3$, since the order of γ_0 is equal to 3.

(ii) Suppose that S_1 contains at least three elements. If we define μ_0 as above, it follows that $\|\mu\| \geq \|\mu_0\| \geq ((17)^{1/2} + 1)/4$ by the hypothesis of induction, since the number of the elements of $S(\mu_0) = S_1 - \gamma_1$ decreases at least 1 from that of S .

(II) Suppose that S contains no coset of any nontrivial cyclic subgroup of Γ . In this case, we have the following two possibilities, (iii) and (iv):

(iii) Assume that there exists a nonzero γ_0 in Γ such that S contains all of the elements $\gamma + (p+1)\gamma_0$, $\gamma + (p+2)\gamma_0$ and $\gamma + (p+3)\gamma_0$ for some γ in Γ and for some integer p . Without loss of generality, we may then clearly assume that $\gamma = 0$ and $p = -1$ (if necessary, consider $S - \gamma - (p+1)\gamma_0$ in place of S). Let $\{B_j\}$ be the collection of the disjoint subsets of S each of which consists of the elements $(q_j + l)\gamma_0$, $1 \leq l \leq n_j$, where n_j and q_j are integers such that neither

$q_j\gamma_0$ nor $(q_j+n_j+1)\gamma_0$ is in S . Our assumption implies that at least one of the n_j is larger than 2.

First assume that some n_j are odd. Putting $n_j=2k+1$, we define $u=(q_j+1)\gamma_0$, $v=(q_j+k+1)\gamma_0$, $w=(k+1)\gamma_0$ if $k\geq 1$, and set $u=0$, $v=(q_j+1)\gamma_0$, $w=\gamma_0$ if $k=0$. It is easy to show that so defined u , v , and w satisfy the conditions in the lemma, and we have the desired conclusion.

Next assume that all n_j are even. Put $n_j=2k_j$. If there exists an $n_j\geq 4$ such that $(q_j+n_j+2)\gamma_0$ is not in S , we define $u=(q_j+1)\gamma_0$, $v=(q_j+k_j+1)\gamma_0$, $w=(k_j+1)\gamma_0$, and appeal to the lemma. If $(q_j+n_j+2)\gamma_0$ are in S , for all $n_j\geq 4$, then either the order of γ_0 is equal to n_1+1 , or $n_i>n_j$ for some n_i and n_j such that $(q_i+n_i+2)\gamma_0=(q_j+1)\gamma_0$. In the latter case we define $u=(q_i+1)\gamma_0$ or $u=(q_i+2)\gamma_0$, and $v=(q_i+k_i+k_j+1)\gamma_0$, $w=(k_i+k_j+1)\gamma_0$. The lemma applies again. In the first case, let μ_0 be the convolution of μ and m_0 , the Haar measure of the annihilator of the cyclic subgroup $\Gamma(\gamma_0)$; then $S(\mu_0)=B_1$ and so

$$\begin{aligned}\|\mu\| &\geq \|\mu_0\| = \int_G \left| \sum_{l=1}^{n_1} (x, (q_1+l)\gamma_0) \right| dm(x) \\ &= \frac{1}{n_1+1} \sum_{r=0}^{n_1} \left| \sum_{l=1}^{n_1} \exp\left(\frac{2\pi l r}{n_1+1} i\right) \right| = \frac{2n_1}{n_1+1} \geq \frac{8}{5}\end{aligned}$$

since $n_1\geq 4$.

(iv) Finally assume that there is no γ_0 as in (iii). Take distinct γ_0 and γ_1 in S which are different from 0. We note that none of $2\gamma_0$, $-\gamma_0$, and $-\gamma_1$ belongs to S . If neither $\gamma_1+\gamma_0$ nor $\gamma_1-\gamma_0$ belongs to S , then the lemma obviously applies to our μ . If $\gamma_1+\gamma_0$ is in S , then $\gamma_1+2\gamma_0$ is not in S by our assumption, and so we define $u=0$, $v=\gamma_0$, $w=\gamma_1+\gamma_0$ in the lemma. If $\gamma_1-\gamma_0$ is in S , then neither $\gamma_0+\gamma_1$ nor $\gamma_0-\gamma_1=-(\gamma_1-\gamma_0)$ is in S ; put $u=0$, $v=\gamma_0$, $w=\gamma_1$ and appeal to the lemma again.

The proof of Theorem 2 is now completed.

The following corollary is an easy conclusion of Theorem 1.

COROLLARY. *Every μ with $1 < \|\mu\| < ((17)^{1/2}+1)/4$ has norm equal to one of the values*

$$2/(p \tan(\pi/2p)), \quad 4/\pi, \quad 2/(q \sin(\pi/2q)),$$

where p ranges over the set of all the even integers larger than 2 and q over the set of all the odd integers larger than 7. In particular, we have $\|\mu\| \geq (2^{1/2}+1)/2$ if $\|\mu\| > 1$.

We do not know whether the constant $((17)^{1/2}+1)/4$ is the best possible one in Theorem 2. However, an example shows that it cannot be increased beyond $4/3$. In Theorem 2, let $S = \{0, \gamma, 2\gamma\}$, where γ is an element in Γ with order 6, then we have $\|\mu\| = 4/3$.

REFERENCES

1. T. Ito and I. Amemiya, *A simple proof of the theorem of P. J. Cohen*, Bull. Amer. Math. Math. Soc. **70** (1964), 774–778.
2. P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960) 191–212.
3. W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.
4. ———, *Idempotent measures on abelian groups*, Pacific J. Math. **9** (1959), 195–209.

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