SOME IRREGULAR DISTRIBUTIONS MODULO 1

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Let us denote by $r(x)$ the distance from the real number x to the nearest integer so that $0 \leq r(x) \leq \frac{1}{2}$ and for any T , $\int_T^{T+1} r(x) dx = \frac{1}{4}$. Let $0 < y < 1$, $0 < \alpha < \frac{1}{2}$. For $n = 1, 2, 3, \dots$, define $B(n)$ to be the number of integers $k \geq 1$ for which $r(2^{-k}ny) > \alpha$. It is to be expected that, as $n \rightarrow \infty$,

$$B(n) \sim (1 - 2\alpha) \log n / \log 2.$$

But in fact

THEOREM. For any $b \in (0, 1)$, $\sum_{n=1}^{\infty} b^{B(n)} = +\infty$.

The proof is based upon the properties of a random Fourier-Stieltjes transform. Let p be a fixed number in $(0, 1)$ and $X_1, X_2, \dots, X_n, \dots$ independent random variables with $P\{X_n = 1\} = p$, $P\{X_n = 0\} = 1 - p$, $1 \leq n < \infty$. Each vector $X = (X_1, \dots, X_n)$ determines a Fourier-Stieltjes transform $\phi(u)$, $-\infty < u < \infty$, by the formula

$$\phi(u) = \prod_{m=1}^{\infty} \cos(2^{-m}\pi u X_m) = \prod_{X_m=1} \cos(2^{-m}\pi u).$$

Here the transform ψ of a measure μ is defined

$$\psi(u) \equiv \int_{-\infty}^{\infty} e^{-iut} \mu(dt).$$

Observe that if μ is concentrated in $[-\pi, \pi]$, then ψ is, as defined, an entire analytic function and $|\psi(u)| \leq \|\mu\| e^{\pi|u|}$, $u \in \mathbb{R}^2$. With the definition of ϕ , we see that it is the transform of a measure in the set $E(X)$ of all sums

$$t(X, \epsilon) = (\pi/2)[\epsilon_1 X_1 + \dots + 2^{-m+1} \epsilon_m X_m + \dots]$$

where $\epsilon_m = \pm 1$ ($1 \leq m < \infty$). For the construction of this measure (the L -measure of $E(X)$) cf. Kahane and Salem [1, Part I].

LEMMA 1. For almost all X ,

$$\liminf_{j \rightarrow \infty} \max_{t \in E(X)} |\sin(2^j t)| = 0.$$

PROOF. Let $n_j = 2^j$ ($1 \leq j < \infty$). Now if $X_m = 0$ for $j < m \leq j+l$ ($l > 1$), then

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$$r\left(\frac{1}{\pi} n_j t\right) \leq \sum_{m>j+l}^{\infty} 2^{j-m} = \sum_{m>l}^{\infty} 2^{-m} = 2^{-l}.$$

Now $X_m = 0$ for $j < m \leq j+l$ with probability $(1-p)^l > 0$, so that this occurs infinitely often, for example, if j runs through the values $l+2$, $2(l+2)$, $\dots$, (Borel-Cantelli). Hence

$$\liminf_{j \rightarrow \infty} \max \left| \sin(n_j t) \right| \leq 2^{-l} \pi.$$

Since l was arbitrary $\liminf = 0$, almost surely, as required.

COROLLARY. $\liminf_{j \rightarrow \infty} |\phi(2^{j+1}) - 1| = 0$, *almost surely*.

PROOF. $\phi(2^{j+1}) = \int_{E(X)} \cos(2^{j+1}t) \mu(dt)$ for a probability μ concentrated in $E(X)$, and the identity $\cos(2^{j+1}t) = 1 - 2 \sin^2(2^j t)$ completes the proof. In fact, μ is just the L -measure of $E(X)$.

This is an adaptation of a method of R. Salem [1, p. 91].

COROLLARY. $\limsup_{n \rightarrow \infty} |\phi(ny)| \geq C(y) > 0$, *almost surely, for some positive constant $C(y)$ independent of p* .

PROOF. According to a theorem of M. L. Cartwright, every entire function F whose growth is regulated by the bound $|F(z)| \leq Ae^{\pi|z|}$ allows an estimate

$$C(y) \sup_{-\infty < x < \infty} |F(x)| \leq \sup_{-\infty < n < \infty} |F(ny)|.$$

It is easy to replace "sup" by "lim sup" in the proof; see Pflüger [2, pp. 312–314]. The function ϕ clearly has the required growth.

We can now complete the proof. Since, by the Corollary above,

$$\limsup_{n \rightarrow \infty} \prod_{m=1}^n |\cos(\pi n 2^{-m} y X_m)| \geq C(y) > 0 \quad \text{almost surely,}$$

$$\liminf_{n \rightarrow \infty} \sum P\{X_m : r(n 2^{-m} y) > \alpha\} < D(y) \quad \text{almost surely,}$$

where $D(y)$ depends on α and y but *not* on p . We may specify $D(y) \geq 1$. Hence

$$\sum_{n=1}^{\infty} P\{\sum \{X_m : r(n 2^{-m} y) > \alpha\} < D(y)\} = \infty.$$

Now $\sum \{X_m : r(n 2^{-m} y) > \alpha\}$ has the same distribution as $\sum_{m=1}^{B(n)} X_m$.

Let $D(y)B^*(n) \leq B(n) < D(y)(B^*(n)+1)$ for integers $B^*(n)$, whence

$$P\left\{\sum_{m=1}^{B(n)} X_m < D(y)\right\} \leq D(y)P\left\{\sum_{m=1}^{B^*(n)} X_m = 0\right\} = D(y)(1-p)^{B^*(n)}.$$

Setting $0 < p = 1 - b^{D(y)} < 1$, we find

$$\sum_{n=1}^{\infty} b^{B(n)} = \sum_{n=1}^{\infty} (1 - p)^{D(y)B^*(n)} = \infty.$$

REFERENCES

1. J.-P. Kahane and R. Salem, *Ensembles parfaits et series trigonometriques*, Actualités Sci. Indust., No. 1301, Hermann, Paris, 1963.
2. A. Pflüger, *On functions bounded at the lattice points*, Proc. London Math. Soc. (2) **42** (1936–1937), 305–315.

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