A CLASS OF LINEAR TRANSFORMATIONS WHICH CAN BE WRITTEN AS THE PRODUCT OF PROJECTIONS

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1. Introduction. Over the years there have been many results concerning the structure of linear operators on a complex linear space; most notable among them being the spectral type representations. In this paper we present a factorization type decomposition of certain singular linear operators into the product of a finite number of projections. The easiest stated result in this direction being: An $n \times n$ matrix is singular if and only if it can be written as the product of a finite number of proper projections.

2. Main results. In what follows we shall let X denote a linear space over the real or complex field and by $\nu(T)$ the Hamel dimension of the null space of a linear operator T which maps X into X . A projection P of X into X will be said to be a proper projection if and only if $\nu(P) \geq 1$, i.e. P not the identity operator.

LEMMA 1. *Let $N: X \rightarrow X$ be a linear operator which is nilpotent of index q and X_1 a linear subspace of X which has a Hamel basis of the form*

$$\{x_\alpha, Nx_\alpha, N^2x_\alpha, \dots, N^{q-1}x_\alpha: \alpha \in \mathfrak{A}\}$$

where $\mathfrak{A}$ is some indexing set. Then there exists a linear subspace X_2 which is also invariant under N and for which $X = X_1 \oplus X_2$.

The proof is by induction on the index q . When $q=1$ the lemma is trivially true. Assume that the lemma is true for $q-1$. Observe that the restriction $N|_R$ of N to its range R is nilpotent of index $q-1$ and that the linear subspace $M_0 = R \cap X_1$ is the linear span of vectors of the form

$$\{y_\alpha, Ny_\alpha, \dots, N^{q-2}y_\alpha: y_\alpha = Nx_\alpha, \alpha \in \mathfrak{A}\}.$$

By the induction hypothesis there exists a linear subspace Y_0 which is invariant under N and $R = M_0 \oplus Y_0$. Letting $Y_1 = \{x: Nx \in Y_0\}$ one writes $Nx = y + z$ with $y \in M_0$ and $z \in Y_0$. Since

$$y = \sum_{j=1}^n \left\{ \sum_{i=1}^{q-1} a_{ij} N^i x_{\alpha_j} \right\} = N \left(\sum_{j=1}^n \sum_{i=0}^{q-2} a_{i+1,j} N^i x_{\alpha_j} \right) = Ny_0 \text{ with } y_0 \in X_1,$$

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it follows that $N(x - y_0) = z$; that is $(x - y_0) \in Y_1$. Thus $x = y_0 + (x - y_0)$ with $y_0 \in X_1$ and $(x - y_0) \in Y_1$ which establishes that $X = X_1 + Y_1$. If $x \in X_1 \cap Y_0$ then $Nx \in M_0$ and $Nx \in Y_0$. Thus $Nx = 0$ and since $x \in X_1$,

$$x = \sum_{j=1}^m \left\{ \sum_{i=0}^{q-1} b_{ij} N^i x_{\alpha_j} \right\}$$

and therefore

$$0 = Nx = \sum_{j=1}^m \left\{ \sum_{i=1}^{q-1} b_{i-1j} N^i x_{\alpha_j} \right\}.$$

From the linear independence of $\{N^i x_{\alpha_j}\}$ it follows that $b_{ij} = 0$ for $i = 0, 1, \dots, q-2, j = 1, 2, \dots, m$ so that

$$x = \sum_{j=1}^m b_{q-1j} N^{q-1} x_{\alpha_j}$$

implying that $x \in M_0$. Thus we have proven that $X_1 \cap Y_0 \subset M_0 \cap Y_0$, hence $x = 0$.

Now recall that $X = X_1 + Y_1$, $Y_0 \subset Y_1$, $X_1 \cap Y_1 \subset Y_1$ and $Y_0 \cap (X_1 \cap Y_1) = 0$. Let Z be a linear subspace such that $Y_1 = Z \oplus Y_0 \oplus \{X_1 \cap Y_1\}$ and define $X_2 = Z \oplus Y_0$. Observe that $X_2 \subset Y_1$, $X_2 \cap (X_1 \cap Y_1) = 0$ implies that $X_1 \cap X_2 = 0$, showing that $X = X_1 \oplus X_2$. Since $X_2 \subset Y_1$, $NX_2 \subset Y_0 \subset X_2$ showing that X_2 is invariant under N .

LEMMA 2. *If $N: X \rightarrow X$ is a linear mapping which is nilpotent of index q then there exist invariant linear subspaces*

$$Y_{q_1}, Y_{q_2}, \dots, Y_{q_j}, \quad 0 < q_1 < q_2 < \dots < q_j = q$$

of X such that $X = Y_{q_1} \oplus Y_{q_2} \oplus \dots \oplus Y_{q_j}$, with each subspace Y_{q_k} having a Hamel basis of the form

$$\{x_\alpha, Nx_\alpha, \dots, N^{q_k-1}x_\alpha; \alpha \in \mathfrak{A}_k\} \quad (k = 1, 2, \dots, j).$$

Thus if E_k denotes the projection $Ex = x_k$, if $x = x_1 + x_2 + \dots + x_j$ where $x_k \in Y_{q_k}$, then NE_k is nilpotent of index q_k , $k = 1, 2, \dots, j$.

PROOF. Let $\mathfrak{X}$ be the family of all linear subspaces of X which have a Hamel basis of the form $\{x_\alpha, Nx_\alpha, \dots, N^{q_j-1}x_\alpha\}$. Partially ordering this family by inclusion and applying Zorn's lemma establishes the existence of a maximal element Y_{q_j} in $\mathfrak{X}$. Lemma 1 guarantees the existence of a linear subspace M which is invariant under N and such that $X = M \oplus Y_{q_j}$. Because of the maximality of Y_{q_j} , N is nilpotent

of index q_{j-1} on M with $q_{j-1} < q_j$. Repeated application of this argument gives the desired result.

THEOREM 1. *If $N: X \rightarrow X$ is a linear nilpotent operator of index q then N can be written as the product of a finite number of projections.*

PROOF. Let $X = Y_{q_1} \oplus Y_{q_2} \oplus \cdots \oplus Y_{q_j}$ be a decomposition of X of the type discussed in Lemma 2. Relative to a Hamel basis for Y_{q_k} of the form $\{x_\alpha, Nx_\alpha, \dots, N^{q_k-1}x_\alpha: \alpha \in \mathfrak{A}_k\}$ define $P_{q_k}^{(i)}$ ($i=1, 2, \dots, q_k-1$) to be the projection on

$$Y_{q_1} \oplus \cdots \oplus Y_{q_{k-1}} \oplus \text{span}\{x_\alpha, \dots, N^{i-2}x_\alpha, N^i x_\alpha, \dots, N^{q_k-1}x_\alpha: \alpha \in \mathfrak{A}_k\} \\ \oplus Y_{q_{k+1}} \oplus \cdots \oplus Y_{q_j}$$

along the span $\{N^{i-1}x_\alpha - N^i x_\alpha: \alpha \in \mathfrak{A}_k\}$. That is, $P_{q_k}^{(i)}$ ($i=1, 2, \dots, q_k-1$) is the linear transformation which transforms $N^{i-1}x_\alpha$ into $N^i x_\alpha$ for each $\alpha \in \mathfrak{A}_k$ and the remaining basis elements into themselves. Also define $P_{q_k}^{(q_k)}$ to be the projection on

$$Y_{q_1} \oplus \cdots \oplus Y_{q_{k-1}} \oplus \text{span}\{x_\alpha, Nx_\alpha, \dots, N^{q_k-2}x_\alpha: \alpha \in \mathfrak{A}_k\} \\ \oplus Y_{q_{k+1}} \oplus \cdots \oplus Y_{q_j} \text{ along } \{N^{q_k-1}x_\alpha: \alpha \in \mathfrak{A}_k\}.$$

Then $N = \prod_{k=1}^j \left\{ \prod_{i=1}^{q_k} P_{q_k}^{(i)} \right\}$.

THEOREM 2. *Let $E_i: X \rightarrow X$, $i=1, 2$, be two proper projections such that $E_i E_j = 0$ whenever $i \neq j$ and $I = E_1 + E_2$. Let $N: X \rightarrow X$ be a linear nilpotent operator for which $E_i N = N E_i$, $i=1, 2$, and the linear subspace $E_2 X$ has a Hamel basis of the form*

$$\{y_\beta, N y_\beta, \dots, N^{q-1} y_\beta: \beta \in \mathfrak{B}\}.$$

If $T = \lambda E_2 + N$, $\lambda \neq 0$ and the cardinal of $\mathfrak{B} \leq \nu(T)$ then T can be written as the product of a finite number of projections.

PROOF. The subspaces $X_1 = E_1 X$ and $X_2 = E_2 X$ are invariant under T and $X = X_1 \oplus X_2$. Since $T E_1$ is nilpotent it can be written as the finite product of projections

$$T E_1 = \left\{ \prod_{i=1}^m P_i \right\} E_1$$

where P_i , $i=1, 2, \dots, m$ is a projection which leaves X_2 invariant. Let $\{x_\alpha: \alpha \in \mathfrak{A}\}$ be a Hamel basis for the null space of T and extend this linearly independent set to a Hamel basis for X_1 . Since $\text{card } \mathfrak{B} \leq \text{card } \mathfrak{A}$, there exists a function f which maps $\mathfrak{B}$ into $\mathfrak{A}$ in a one to one manner. Now define the projections:

(i) $Q_1: x_{f(\beta)} \rightarrow \lambda N^{q-1} y_\beta$ for $\beta \in \mathfrak{B}$ and the remaining basis entries remaining invariant,

(ii) $Q_2: N^{q-1} y_\beta \rightarrow x_{f(\beta)}$ for $\beta \in \mathfrak{B}$ and the remaining basis entries remaining invariant.

Also for $k=0, 1, \dots, q-2$ define

(iii) $V_1^{(k)}: x_{f(\beta)} \rightarrow \lambda N^k y_\beta$ for $\beta \in \mathfrak{B}$ and the remaining basis entries remaining invariant,

(iv) $V_2^{(k)}: N^k y_\beta \rightarrow N^{k+1} y_\beta$ for $\beta \in \mathfrak{B}$ and the remaining basis entries remaining invariant,

(v) $V_3^{(k)}: x_\alpha \rightarrow 0$ for $\alpha \in \mathfrak{A}$ and $N^k y_\beta \rightarrow N^k y_\beta + x_{f(\beta)}$ for $\beta \in \mathfrak{B}$ with the remaining basis entries remaining invariant.

(vi) $P: x_\alpha \rightarrow 0$ for $\alpha \in \mathfrak{A}$ and the remaining basis entries remaining invariant.

Then $T = \left\{ \prod_{i=1}^m P_i \right\} \left\{ \prod_{k=0}^{q-2} V_1^{(k)} V_2^{(k)} V_3^{(k)} \right\} Q_1 Q_2 P$.

THEOREM 3. Let $E_i: X \rightarrow X$, $i=1, 2, \dots, k$ be proper projections such that $I = E_1 + E_2 + \dots + E_k$ and $E_i E_j = 0$ whenever $i \neq j$ and let $N: X \rightarrow X$ be a linear nilpotent operator of index q for which $E_i N = N E_i$ for $i=1, 2, \dots, k$. If

$$T = \sum_{i=2}^k \lambda_i E_i + N \quad (\lambda_i \neq 0, i = 2, 3, \dots, k)$$

and

$$(1) \quad -\left\{ \nu(N^i E_j) - 2\nu(N^{i-1} E_j) + \nu(N^{i-2} E_j) \right\} \leq \nu(T),$$

$$j = 2, 3, \dots, k, i = 2, 3, \dots, q+1,$$

then T can be written as the product of a finite number of projections.

PROOF. Under either T or N the subspaces $X_i = E_i X$, $i=1, 2, \dots, k$, are invariant and $X = X_1 \oplus X_2 \oplus \dots \oplus X_k$. On X_1 , T is nilpotent and hence $T E_1$ can be written as the finite product of projections

$$T E_1 = \left\{ \prod_{i=1}^m P_i \right\} E_1$$

where P_i , $i=1, 2, \dots, m$, are projections which leave $X_2 \oplus \dots \oplus X_k$ invariant. Lemma 2 implies that there exist subspaces $Y_{p_1}, Y_{p_2}, \dots, Y_{p_j}$, $0 < p_1 < \dots < p_j \leq q$, such that $X_2 = Y_{p_1} \oplus Y_{p_2} \oplus \dots \oplus Y_{p_j}$ with each subspace Y_{p_i} having a Hamel basis of the form

$$\{x_\alpha, N x_\alpha, \dots, N^{p_i-1} x_\alpha: \alpha \in \mathfrak{A}_i\}, \quad i = 1, 2, \dots, j.$$

Condition (1) implies that cardinal of $\mathfrak{A}_i \leq \nu(T)$ for $i=1, 2, \dots, j$ and, using a construction similar to that which was used in the proof

of Theorem 2, there exists for each subspace Y_{p_i} a finite product of projections defined on X which equals T on Y_{p_i} and leaves the remaining subspaces invariant. The product of these products equals T on X_2 and leaves the span of $X_1 \cup X_3 \cup \cdots \cup X_k$ invariant. In a similar manner, there exists a finite product of projections which equals T on X_i ($i=3, \cdots, k$) and leaves the other invariant subspaces unaltered. The product of all the projections described above equals T on X .

Let X be a complex Banach space and let $T: X \rightarrow X$ be a continuous linear operator whose resolvent operator $R(\lambda; T)$ is rational; that is, $R(\lambda, T) = P(\lambda)/Q(\lambda)$ where P is a polynomial with coefficients in X and q is a polynomial with complex coefficients. Denoting the distinct poles of $R(\lambda, T)$ by $\lambda_1, \lambda_2, \cdots, \lambda_k$ with orders $m_1, m_2, \cdots, m_k$ respectively and letting M_i be the null space of $(T - \lambda_i I)^{m_i}$, $i=1, 2, \cdots, k$ then T is completely reduced by $(M_1, M_2, \cdots, M_k)$, (see Dunford and Schwartz [1, Chapter 7]). Then letting E_i be the projection $E_i x = x_i$, $x = x_1 + x_2 + \cdots + x_k$, $x_i \in M_i$, $i=1, 2, \cdots, k$, the operator T can be written as

$$(2) \quad T = \sum_{i=1}^k \lambda_i E_i + N$$

where N is the nilpotent operator $N = \sum (T - \lambda_i I) E_i$ of index $q = \max \{m_1, m_2, \cdots, m_k\}$. Thus if T also satisfies condition (1) then according to the previous theorem, T can be written as the finite product of projections. These projections, however, need not be continuous. They will be continuous if and only if their range and null spaces are closed. The following corollary gives an important example when this will always be the case.

COROLLARY 1. *Let X be a complex finite dimensional linear space. Then a linear transformation $T: X \rightarrow X$ is singular if and only if T can be written as the product of a finite number of proper projections.*

PROOF. Assume that T is a singular linear operator and X is n -dimensional. Then there exists a basis $\{x_1, x_2, \cdots, x_n\}$ for X and a set of integers $0 = n_0 < n_1 < \cdots < n_k = n$ such that $Tx_i = \lambda_j x_i + x_{i+1}$ for $n_{j-1} < i < n_j$ and $Tx_{n_j} = \lambda_j x_{n_j}$, where $\lambda_j \in \sigma(T)$ with possible repetition and $\lambda_1 = 0$. The subspaces $X_j = \text{span of } \{x_i: n_{j-1} < i \leq n_j\}$ are invariant under T and $T = \sum_{i=1}^k \lambda_i E_i + N$ where N is nilpotent satisfying $Nx_i = x_{i+1}$ for $n_{j-1} < i < n_j$ and $Nx_{n_j} = 0$ and E_j the projection on X_j along $X_1 \oplus \cdots \oplus X_{j-1} \oplus X_{j+1} \oplus \cdots \oplus X_k$. The conclusion follows

immediately from Theorem 3. Conversely if T is the product of a finite number of proper projections then T is singular.

COROLLARY 2. *Let X be a complex Banach space and let $T: X \rightarrow X$ be a continuous linear operator whose range is a proper finite dimensional subspace of X . Then T can be written as the product of a finite number of projections.*

PROOF. Corollary 1 can be applied if X is finite dimensional. The resolvent of T remains rational if X is not of finite dimension. In fact, the resolvent of T is $R(\lambda, T) = \lambda^{-1} \{I + R(\lambda, T_0)T\}$ where $R(\lambda, T_0)$ is the resolvent of the restriction T_0 of T to the range of T . If $p(\lambda) = \lambda^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$ is the minimal polynomial for T then T can be written as $T = \sum_{i=1}^k \lambda_i E_i + N$ where E_i and N satisfy the same conditions as those described in (2). T being a bounded linear transformation with a finite dimensional range implies that T is completely continuous. Thus for $i \neq 1$, the null space of $(\lambda_i I - T)^{m_i}$ is of finite dimension and condition (1) of Theorem 3 is satisfied.

An important type of operator with a finite dimensional range is an integral operator $Tx = \int_a^b k(s, t)x(t)dt$ with a degenerate kernel $k(s, t) = \sum_{i=1}^n \alpha_i(s)\beta_i(t)$.

3. An application. The iterative scheme

$$(3) \quad x_{k+1} = (I - C_k A)x_k + C_k b, \quad k = 0, 1, 2, \dots,$$

for solving the linear system $Ax = b$, where A is assumed to be an $n \times n$ nonsingular matrix, is called a projective iterative method if and only if each $P_k = I - C_k A$, $k = 0, 1, 2, \dots$, is a proper projection. Many common iterative methods can be viewed as a projective iterative method, such as the point and block Gauss Seidel iterative methods and the method of steepest descent. For further examples see Householder and Bauer [3], Gastinel [2] and Kaczmarz [4]. Corollary 1 allows us to determine which iterative schemes of the form (3) can be written as a projective iterative scheme.

THEOREM 4. *The iteration*

$$x_{k+1} = (I - C_k A)x_k + C_k b, \quad k = 0, 1, 2, \dots,$$

can be viewed as a projective iterative scheme if $I - C_k A$ is singular for each $k \geq 0$.

PROOF. Since $I - C_k A$ is singular, projections $P_0, P_1, \dots, P_m$ exist such that $I - C_k A = \prod_{i=0}^m P_i$ and $C_k = \{I - \prod_{i=0}^m P_i\}A^{-1}$. Then the

next iterate x_{k+1} can be obtained from x_k by a finite number of projective type iterations of the form

$$y_{i+1} = P_i y_i + (I - P_i) A^{-1} b \quad \text{for } i = 0, 1, \dots, m$$

with $y_0 = x_k$. It is an easy matter to check that $y_{m+1} = x_{k+1}$.

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