

# EMBEDDING A $p$ -SPHERE IN A PRODUCT OF $p$ -SPHERES

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**1. Introduction.** We will consider 2 embeddings of  $S^p$  in  $S^p \times S^p$ , say  $S_0^p$  and  $S_1^p$ , to be different if we cannot find a self-homeomorphism  $f$  of  $S^p \times S^p$  such that  $f(S_0^p) = S_1^p$ . It is well known that if  $p=1$ , then there are only 2 different embeddings, one which does not separate  $S^1 \times S^1$ , and one which does separate. The proof is attributed to H. Hopf. We shall work simultaneously in the differential and the piecewise linear categories, except when  $p=3$ . In that case, our results will be valid in the PL (piecewise linear) category. The main tool, used in this paper, is the classification of which automorphisms of  $H_p(S^p \times S^p, Z)$  can be realized by self-homeomorphisms of  $S^p \times S^p$ , which was proved in [1].

**2. Self-homeomorphisms of  $S^p \times S^p$ .** All homology groups will have the integers as their coefficient group. A *preferred basis*,  $\{z_1, z_2\}$  for  $H_p(S^p \times S^p)$  is one in which  $z_1$  represents  $S^p \times y$  and  $z_2$  represents  $x \times S^p$ . If  $g$  is an automorphism of  $H_p(S^p \times S^p)$ , then  $g$  can be represented by a  $2 \times 2$  matrix with integral entries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with  $|ad-bc|=1$ , where it will be understood that

$$g(z_1) = az_1 + cz_2 \quad \text{and} \quad g(z_2) = bz_1 + dz_2.$$

We shall denote the above group of matrices as  $G$ .

The subgroup of  $G$ , where

$$a \equiv d \pmod{2} \quad b \equiv c \pmod{2},$$

is denoted by  $G'$ , and the subgroup of  $G$  where the entries are either 1,  $-1$  or 0 is denoted by  $G''$ .

**THEOREM 1.** *A necessary and sufficient condition that a given automorphism  $g$  of  $H_p(S^p \times S^p)$  can be induced by a self-homeomorphism of  $S^p \times S^p$  is that*

1.  $g \in G$  when  $p=1, 3, 7$ .
2.  $g \in G'$  when  $p$  is odd and not equal to 1, 3, 7.
3.  $g \in G''$  when  $p$  is even.

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PROOF. This theorem is implied in [1], although it is not stated therein. The proof for the necessity follows from Lemma 3.3, the sufficiency from Proposition 2.5.

**3. The classification.** In what follows, it will be assumed that  $p \geq 3$ .

Let  $S_0^p$  and  $S_1^p$  be embedded in  $S^p \times S^p$ , such that they represent the same element in  $H_p(S^p \times S^p)$ . In the PL category, there exists a homeomorphism  $f$  of  $S^p \times S^p$  onto itself such that  $f(S_0^p) = S_1^p$ , by [4, Corollary 1 to Theorem 24]. When  $p \geq 4$ , the same result is true in the differential category according to [2]. Now, since any element in  $H_p(S^p \times S^p)$  can be represented by an embedded sphere, it only remains to show which elements are equivalent via a homeomorphism.

**THEOREM 2.** *Any  $S^p$  embedded in  $S^p \times S^p$  is equivalent to one of the following different embeddings.*

1. *A sphere representing  $(i, 0)$ ,  $i = 0, 1, 2, \dots$ , in  $H_p(S^p \times S^p)$  when  $p = 3, 7$ .*
2. *A sphere as above or one which represents  $(i, i)$ ,  $i = 1, 2, \dots$ , when  $p$  is odd and not equal to 3 or 7.*
3. *A sphere representing  $(i, j)$  where  $0 \leq i \leq j$  when  $p$  is even and greater than 2.*

PROOF. Let  $S^p$  be embedded in  $S^p \times S^p$  such that it represents  $(m, n)$  in  $H_p(S^p \times S^p)$ , we shall exclude the case where both  $m$  and  $n$  are 0. Let  $d$  be the greatest common divisor of  $m$  and  $n$ .

*Case 1.* There exist integers  $a$  and  $b$  such that

$$g = \begin{pmatrix} a & b \\ -n/d & m/d \end{pmatrix} \in G$$

and thus, by Theorem 1, we can find a homeomorphism  $f$  such that  $f_* = g$ . Now  $f(S^p)$  represents  $(d, 0)$ . It is easy to see that the above classes are different.

*Case 2.* If either  $m/d$  or  $n/d$  is even, say  $m/d$ , then we can find integers  $a$  and  $b$ , where  $a$  is even, and  $am/d + bn/d = 1$ . Thus

$$g_1 = \begin{pmatrix} a & b \\ -n/d & m/d \end{pmatrix} \in G',$$

and we can find a homeomorphism  $f_1$  such that  $f_1(S^p)$  represents  $(d, 0)$  (if  $n/d$  is even, then  $b$  can be chosen to be even). If both  $m/d$  and  $n/d$  are odd, then we can find integers  $a$  and  $b$  such that  $am/d + bn/d = 1$ . Clearly either  $a$  or  $b$  is even and thus

$$g_2 = \begin{pmatrix} a & b \\ a - n/d & b + m/d \end{pmatrix} \in G'.$$

A sphere which represents  $(i, i)$  is different from one which represents  $(j, 0)$ , since if there existed a homeomorphism which sent one sphere onto the other, then in its matrix, there would be 2 entries  $x$  and  $y$ , in the same row or column, such that  $|x| = |y|$ . Such a matrix cannot belong to  $G'$ .

*Case 3.* By Theorem 1, the only homeomorphisms which can occur, can only interchange  $m$  and  $n$  or multiply one or the other by  $-1$ . Therefore the proof of Theorem 2 is finished.

**4. Questions.** One would certainly like to drop the restrictions on  $p$ . There are several obstructions to doing so. In the PL category, since Theorem 1 holds when  $p=2$ , the problem is whether 2 spheres representing the same element in  $H_2(S^2 \times S^2)$ , are equivalent. It is known [3] that any element in the homology group can be represented by an embedded sphere. In the differential category, one not only has the same problem as in the PL category, but when  $p=2$ , it is not even known which elements in  $H_2(S^2 \times S^2)$  can be represented by embedded spheres. For instance, it is known that  $(2, 2)$  cannot be represented by a differentiably embedded sphere [3]. In the topological category, it seems reasonable to conjecture that the classification for locally flat embedded spheres should be the same as in Theorem 2.

#### REFERENCES

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