A CRITERION FOR n-n OSCILLATIONS IN DIFFERENTIAL EQUATIONS OF ORDER $2n$

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For the second-order equation $y''+qy=0$, Wintner [6] proved that a sufficient condition for oscillation was that

$$t^{-1}\int^{t} q(x)(t-x)dx \to \infty \quad \text{as } t \to \infty. \tag{1}$$

Independently Leighton [2] proved that a sufficient condition for oscillation of $(ry')'+qy=0$ was that q be positive for sufficiently large x and that

$$\int^{\infty} r(x)^{-1}dx = \infty \quad \text{and} \quad \int^{\infty} q(x)dx = \infty. \tag{2}$$

Subsequently Leighton [3] proved that conditions (2) were sufficient without the restriction q be positive for sufficiently large x.

In this paper we prove analogous theorems for the linear equation of order $2n$ $(ry^{(n)})^{(n)}+(-1)^{n-1}qy=0$. There will be no sign restrictions on the function q. For $n=1$, the earlier results of Wintner and Leighton will be contained in our theorems. The results of this paper parallel earlier results of the author for a fourth-order equation with middle term [1].

Throughout r and q denote continuous, real-valued functions on a ray $[a, \infty)$ and r is assumed to be positive-valued. If the real-valued function y has n continuous derivatives and $ry^{(n)}$ has n continuous derivatives, then we define $L(y)$ by:

$$L(y) = (ry^{(n)})^{(n)} + (-1)^{n-1}qy. \tag{3}$$

Such a y is said to be *admissible* for L.

The operator L is called *oscillatory* on $[a, b]$ if and only if there is an admissible function y, $y\not\equiv 0$, and numbers c and d, $a\leqq c<d\leqq b$, such that $L(y)=0$ and

$$y(c) = \cdots = y^{(n-1)}(c) = 0 = y(d) = \cdots = y^{(n-1)}(d). \tag{4}$$

Otherwise L is called *nonoscillatory* on $[a, b]$.

For $b>a$, let $\mathfrak{A}(b)$ denote the set of all real-valued y on $[a, b]$ such

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that (a) y has $n-1$ continuous derivatives on $[a, b]$ with $y^{(n-1)}$ absolutely continuous, (b) $y^{(n)}$ is essentially bounded ($y^{(n)}$ denotes the almost everywhere derivative of $y^{(n-1)}$) and (c) y satisfies the boundary conditions (4) with $a=c$ and $b=d$.

The function I_b is defined on $\mathfrak{A}(b)$ by:

$$I_b(y) = \int_a^b [r(x)y^{(n)}(x)^2 - q(x)y(x)^2]dx. \tag{5}$$

For our basic criterion of oscillation we consider a vector-matrix formulation of $L(y)=0$. Let the $n\times n$ matrices of functions $A=[a_{ij}]$, $B=[b_{ij}]$ and $C=[c_{ij}]$ be defined by:

$$\begin{aligned} a_{ij} &= 0 \quad \text{if } j-i\neq 1, \qquad & b_{ij} &= 0 \quad \text{if } i\neq n \text{ or } j\neq n \\ &= 1 \quad \text{if } j-i=1, & &= 1/r \quad \text{if } i=n \text{ and } j=n \end{aligned}$$

and

$$\begin{aligned} c_{ij} &= 0 \quad && \text{if } i\neq 1 \text{ or } j\neq 1, \\ &= -q && \text{if } i=1 \text{ and } j=1. \end{aligned}$$

Then if $L(y)=0$,

$$\eta = [y^{(i-1)}]_{i=1}^n \quad\text{and}\quad \xi = [(-1)^{n-i}(ry^{(n)})^{(n-i)}]_{i=1}^n, \tag{6}$$

it is readily verified that

$$\eta' = A\eta + B\xi, \qquad \xi' = C\eta - A^T\xi \tag{7}$$

where A^T denotes the transpose of A.

Conversely, if (η, ξ) is a pair of absolutely continuous real vector-valued functions on $[a, b]$ such that (7) hold almost everywhere, then it follows that (7) hold everywhere, and the first component η_1 of η is admissible for L and $L(\eta_1)=0$.

Reid [5, p. 673] has defined the system (7) to be *oscillatory* on $[a, b]$ if and only if there is a pair (η, ξ) of absolutely continuous real or complex vector-valued functions on $[a, b]$ such that (7) hold almost everywhere, $\eta\not\equiv 0$, and there are numbers c and d, $a\leqq c<d\leqq b$, such that $\eta(c)=0=\eta(d)$. The one-to-one correspondence between solutions y of $L(y)=0$ and the first components η_1 of solutions of (7) proves that L is oscillatory on $[a, b]$ if and only if (7) is oscillatory on $[a, b]$.

We remark that in the terminology of Reid [5, p. 673], the system (7) is identically normal on every subinterval of $[a, \infty)$ since if

(η, ξ) is a solution and $\eta \equiv 0$, then the first equation of (7) implies $\xi_n \equiv 0$, and the second equation of (7) implies successively that $\xi_{n-1} \equiv 0, \cdots, \xi_1 \equiv 0$.

Our basic criterion for oscillation is the following:

THEOREM 1. *If there exists a* $y \in \mathfrak{A}(b)$, $y \not\equiv 0$, *such that* $I_b(y) \leqq 0$, *then* L *is oscillatory on* $[a, b]$.

PROOF. We note first that $C(x)^T = C(x)$, $B(x)^T = B(x)$ and that $B(x)$ is nonnegative definite on $[a, b]$. Moreover, if $y \in \mathfrak{A}(b)$, $\eta = \{y^{(i-1)}\}_{i=1}^n$, $\xi = \{\xi_i\}$ where $\xi_n = ry^{(n)}$ and $\xi_i = 0$ otherwise, then

$$\xi^T B\xi + \eta^T C\eta = r(y^{(n)})^2 - qy^2.$$

Then in the terminology of Reid [5, p. 678], we have $(\eta, \xi) \in \mathfrak{D}_0[a, b]$, $\eta \not\equiv 0$ and $I[\eta, \xi; a, b] = I_b(y) \leqq 0$. Thus by Theorem 5.2 of [5], L is oscillatory on $[a, b]$.

From Theorem 1 we obtain a comparison theorem for oscillation.

THEOREM 2. *If* r_1 *and* q_1 *are continuous, real-valued functions on* $[a, b]$ *with* r_1 *positive-valued*, $y \in \mathfrak{A}(b)$ *is a nontrivial solution of* $L(y) = 0$ *and* $L_1(y) = (r_1 y^{(n)})^{(n)} + (-1)^{n-1} q_1 y$, *then* L_1 *is oscillatory on* $[a, b]$ *if*

$$(8) \qquad \int_a^b [(r(x) - r_1(x))y^{(n)}(x)^2 - (q(x) - q_1(x))y(x)^2]dx \geqq 0.$$

PROOF. Let J_b be defined by the right-hand side of (5) where r and q are replaced by r_1 and q_1 respectively. Then equation (8) reduces to $I_b(y) - J_b(y) \geqq 0$. Integrating $\int_a^b (ry^{(n)}(x)) \cdot y^{(n)}(x)dx$ by parts n times proves that $I_b(y) = (-1)^n \int_a^b L(y) \cdot y(x)dx$. Since $L(y) = 0$, equation (8) is equivalent to $J_b(y) \leqq 0$. Theorem 1 now implies L_1 is oscillatory on $[a, b]$.

As a corollary we have a generalization of the Sturm-Picone Theorem for second-order equations.

COROLLARY 2.1. *If* L *is oscillatory on* $[a, b]$ *and* $r_1(x) \leqq r(x)$ *and* $q_1(x) \geqq q(x)$ *for each* x *in* $[a, b]$, *then* L_1 *is oscillatory on* $[a, b]$.

We now prove our principal theorem.

THEOREM 3. *If there is a positive-valued continuous function* h *on* $[a, \infty)$ *such that as* $t \to \infty$,

(i) $\int_a^t x^{n-1}h(x)dx \to \infty$ *and*

(ii) $\liminf J(t) = -\infty$

where

$$J(t)=\left[\int_a^t\left\{r(x)[(n-1)!h(x)]^2-q(x)\left[\int_x^t(s-x)^{n-1}h(s)ds\right]^2\right\}dx\right]$$
$$\cdot\left[\int_a^t x^{n-1}h(x)dx\right]^{-2},$$

then there is a number $b>a$ such that L is oscillatory on $[a, b]$.

PROOF. For each number $t>a+1$ we construct a function y_t on $[a, t]$ such that $y_t\in\mathfrak{A}(t)$. For some t sufficiently large, we will have $I_t(y_t)<0$, thus proving Theorem 3.

For $t>a+1$, define z_t on $[a, t]$ by

$$z_t(x) = \left[\int_x^t (s-x)^{n-1}h(s)ds\right]\left[\int_a^t s^{n-1}h(s)ds\right]^{-1}.$$

It is clear that for $k=0, \cdots, n-1$,

$$[d^kz_t(x)/dx^k]_{x=t} = 0.$$

For $k=0, \cdots, n-1$, let

$$c_{tk} = [d^kz_t(x)/dx^k]_{x=a+1}.$$

Application of l'Hospital's rule proves that $c_{t0}\to 1$ and for $k=1, \cdots, n-1$, $c_{tk}\to 0$ as $t\to\infty$.

Let p_t be the polynomial

$$p_t(x) = (x-a)^n\sum_{j=0}^{n-1}a_jx^j$$

satisfying, for $k=0, \cdots, n-1$,

$$[d^kp_t(x)/dx^k]_{x=a+1} = c_{tk}. \tag{9}$$

A simple calculation proves that for $k=0, \cdots, n-1$,

$$[d^kp_t(x)/dx^k]_{x=a} = 0.$$

The n coefficients $a_0, \cdots, a_{n-1}$ are determined as solutions to the n linear equations (9). The matrix of coefficients does not depend on t. That the determinant of the matrix of coefficients is nonzero follows from Theorem II of [4] and the fact that p_t is a solution of the differential equation $y^{(2n)}=0$. Hence $a_0, \cdots, a_{n-1}$ are bounded functions of t. Define y' by:

$$y_t(x) = p_t(x) \text{ for } a\leqq x\leqq a+1 \text{ and } y_t(x) = z_t(x) \text{ for } a+1<x\leqq t.$$

Then $y_t\in\mathfrak{A}(t)$.

We have $I_t(y_t) = P(t) + K(t)$, where

$$P(t) = \int_a^{a+1} [r(x)(p_t^{(n)}(x))^2 - q(x)p_t(x)^2]dx$$

and

$$K(t) = \int_{a+1}^t [r(x)(z_t^{(n)}(x))^2 - q(x)z_t(x)^2]dx.$$

Since $a_0, \cdots, a_{n-1}$ are bounded functions of t, $P(t) = O(1)$ as $t \to \infty$. If M is a bound for r, q and h on $[a, a+1]$, then

$$\left| \int_a^{a+1} \left\{ r(x)[(n-1)!h(x)]^2 - q(x)\left[\int_x^t (s-x)^{n-1}h(s)ds\right]^2 \right\} dx \right|$$
$$\leqq M\left\{[(n-1)!M]^2 + \left[\int_a^t (s-a)^{n-1}h(s)ds\right]^2\right\}.$$

Hence as $t \to \infty$,

$$\int_a^{a+1} [r(z_t^{(n)}(x))^2 - q(x)z_t(x)^2]dx = O(1).$$

Thus condition (ii) implies that lim inf $K(t) = -\infty$ as $t \to \infty$. Hence lim inf $I_t(y_t) = -\infty$ as $t \to \infty$. In particular, $I_t(y_t) < 0$ for some sufficiently large t, thus proving Theorem 3.

For $h \equiv 1$, we have a useful corollary of Theorem 3.

COROLLARY 3.1. *If*

$$\limsup_{t\to\infty} t^{-2n} \int_a^t r(x)dx < \infty,$$

and

$$\lim_{t\to\infty} t^{-2n} \int_a^t q(x)(t-x)^{2n}dx = \infty,$$

then for some $b > a$, L is oscillatory on $[a, b]$.

A weaker but more applicable version of Theorem 3 may be stated as follows:

THEOREM 4. *If there is a positive-valued continuous function h on $[a, \infty)$ such that as $t \to \infty$*

(i) $\int_a^t x^{n-1}h(x)dx \to \infty$,

(ii) $\limsup\{\int_a^t r(x)h(x)^2dx\}\{\int_a^t x^{n-1}h(x)dx\}^{-2} < \infty$ *and*

(iii) $t^{1-n}\int_a^t q(x)(t-x)^{n-1}dx \to \infty$,
then there is a number $b>a$ *such that* L *is oscillatory on* $[a, b]$.

PROOF. Our proof will consist of proving that (i), (ii) and (iii) of Theorem 4 imply (ii) of Theorem 3. First we prove two lemmas.

LEMMA 1. *If* f *is a continuous, real-valued function on* $[a, \infty)$ *such that for some integer* $p \geqq 0$,

$$\lim_{t\to\infty} t^{-p}\int_a^t f(x)(t-x)^p dx = \infty,$$

then for each integer $k>p$,

$$\lim_{t\to\infty} t^{-k}\int_a^t f(x)(t-x)^k dx = \infty.$$

PROOF. A straightforward inductive proof using l'Hospital's rule is omitted.

LEMMA 2. *Suppose that* (i) *and* (iii) *of Theorem* 4 *hold, and that for* $i=1, \cdots, n$,

$$Q_i(t) = \int_a^t q(x)\left\{\int_x^t (s-x)^{n-1}h(s)ds\right\}(t-x)^{i-1}dx.$$

Then for $i=1, \cdots, n$ *and as* $t\to\infty$,

$$Q_i(t)/t^{i-1}\int_a^t x^{n-1}h(x)dx \to \infty. \tag{10}$$

PROOF. For $i=1$ we have by (iii),

$$\lim_{t\to\infty} Q_1'(t)/t^{n-1}h(t) = \lim_{t\to\infty} t^{1-n}\int_a^t q(x)(t-x)^{n-1}dx = \infty.$$

Hence (10) holds.

Suppose (10) holds for some i, $1\leqq i<n$. Let M be a positive number. Since

$$Q_{i+1}'(t) = iQ_i(t) + \int_a^t q(x)(t-x)^{n+i-1}h(t)dx,$$

an application of the inductive hypothesis and Lemma 1 yields that for sufficiently large t, say $t\geqq t_0$,

$$Q_{i+1}'(t) \geqq Mit^{i-1}\int_a^t x^{n-1}h(x)dx + Mt^{n+i-1}h(t).$$

Hence for $t > t_0$,

$$Q_{i+1}(t) \geqq Q_{i+1}(t_0) + M\left\{\int_{t_0}^{t}\left[is^{i-1}\int_a^s x^{n-1}h(x)dx + s^{n+i-1}h(s)\right]ds\right\}$$

$$= Q_{i+1}(t_0) + M\left(\int_a^{t_0} x^{n-1}h(x)dx\right)(t^i - t_0^i) + Mt^i\int_{t_0}^{t} x^{n-1}h(x)dx.$$

The above inequality implies

$$\liminf_{t\to\infty} Q_{i+1}(t)/t^i\int_a^t x^{n-1}h(x)dx \geqq M,$$

from which we conclude that (10) holds for $i+1$.

That (ii) of Theorem 3 is a consequence of (i), (ii) and (iii) of Theorem 4 now follows by applying l'Hospital's rule and Lemma 2 to

$$\lim_{t\to\infty}\left\{\int_a^t q(x)\left\{\int_x^t (s-x)^{n-1}h(s)ds\right\}^2 dx\right\}\left\{\int_a^t x^{n-1}h(x)dx\right\}^{-2}$$

$$= \lim_{t\to\infty} Q_n(t)/t^{n-1}\int_a^t x^{n-1}h(x)dx = \infty.$$

For $h = 1/r$, condition (i) implies (ii), and we obtain the following special case of Theorem 4.

COROLLARY 4.1. *If* $\int_a^\infty x^{n-1}r(x)^{-1}dx = \infty$ *and*

$$\lim_{t\to\infty} t^{1-n}\int_a^t q(x)(t-x)^{n-1}dx = \infty,$$

then there is a number $b > a$ *such that* L *is oscillatory on* $[a, b]$.

For $n = 1$, Corollary 4.1 gives the sufficient criterion of Leighton [**3**].

We note that Lemma 1 and Corollary 3.1 imply the previously mentioned result of Wintner for the equation $y'' + qy = 0$. That the condition

$$t^{-2}\int_a^t q(x)(t-x)^2dx \to \infty \text{ as } t \to \infty$$

is more general than Wintner's condition is shown by the following example.

EXAMPLE. Let $w(t) = t^2\sin^2 t$ and let $q = w''$. It then follows that

$$\int_0^t q(x)dx = w'(t) = 2t\sin^2 t + t^2 \sin 2t,$$

$$t^{-1}\int_0^t q(x)(t-x)dx = t\sin^2 t$$

and

$$t^{-2}\int_0^t q(x)(t-x)^2dx = t/3-(1/2-1/4t^2)\sin 2t-(\cos 2t)/2t.$$

Hence by Corollary 3.1, $L(y)=y''+qy$ is oscillatory.

We remark that Theorem 4 is not applicable in this example since condition (iii) does not hold.

References

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