VARIATION CONVERGENCE FOR BERNSTEIN POLYNOMIALS

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1. Introduction. In this paper we investigate the convergence of the variations of Bernstein polynomials. The n th Bernstein polynomial of the real function f is defined to be

$$(1.1) \quad B_n f(x) \equiv \sum_{r=0}^n f(r/n) p_{nr}(x) \quad \text{where} \quad p_{nr}(x) \equiv C_{n,r} x^r (1-x)^{n-r}$$

for $x \in [0, 1]$. We denote the total variation of $B_n f$ by $VB_n f$. It is shown in [3] that for any f and n ,

$$(1.2) \quad VB_n f \leq \sum_{r=0}^{n-1} \left| f\left(\frac{r+1}{n}\right) - f\left(\frac{r}{n}\right) \right| \leq Vf,$$

and also that

THEOREM (1.1). *If f is of bounded variation, then $VB_n f$ converges to the variation of f , computed only over points of continuity of f .*

In view of the fact that $B_n f$ depends only on the rational values of f , we have considered functions f , defined only on the rationals, which we call "skeletons," as in the paper of Herzog and Hill [2]. For functions f and g , we have

$$(1.3) \quad |VB_n f - VB_n g| \leq V(B_n f - B_n g) = VB_n(f - g).$$

Hence we study conditions under which $VB_n f \rightarrow 0$ for a single function.

Herzog and Hill showed that if f satisfies

$$(1.4) \quad \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| = o(n^{1/2})$$

then $B_n f \rightarrow 0$ for each x in $(0, 1)$. Our result is that if f vanishes in a neighborhood of 0 and 1, and if (1.4) holds, then $VB_n f \rightarrow 0$. Furthermore, if the sums are $O(n^{1/2})$, then the $VB_n f$ are bounded. Finally, there exist bounded skeletons such that the above sums are $O(n^{1/2} A_n)$ for an arbitrary monotone sequence $A_n \rightarrow +\infty$ but such that $VB_{n_k} f \rightarrow +\infty$ for a subsequence $\{n_k\}$.

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2. **Preliminary notions.** Most of the facts which will be needed can be found in [2] or [3]. In particular, we need the following:

REMARK (2.1). For all n , $\sum_{r=0}^n p_{nr}(x) = 1$, $B_n f(0) = f(0)$ and $B_n f(1) = f(1)$. (Thus a necessary condition for $\text{VB}_n f \rightarrow 0$ is $f(0) = f(1)$.)

REMARK (2.2). If $\delta > 0$, $x \in [0, 1]$ then

$$(2.1) \quad \sum_{|r/n - x| \geq \delta} p_{nr}(x) \leq \frac{1}{4n\delta^2}.$$

REMARK (2.3). For any $x \in (0, 1)$ and $r = 0, 1, \dots, n$

$$(2.2) \quad p_{nr}(x) \leq C/[nx(1-x)]^{1/2}$$

where C is an absolute constant.

REMARK (2.4). For all n and f

$$(2.3) \quad \text{VB}_n f \leq \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| \text{V} p_{nr}(x) \leq 2 \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| \text{Max } p_{nr}$$

where the Max is taken over $x \in [0, 1]$. The second inequality holds since p_{n0} and p_{nn} are monotone, and p_{nr} , $r = 1, \dots, n-1$ have a single peak at $x = r/n$.

3. A theorem on the convergence of $\text{VB}_n f$.

THEOREM (3.1). Suppose f vanishes off the interval $I_A = [A, 1-A]$ for some $0 < A \leq \frac{1}{2}$. Then if (a)

$$(3.1) \quad \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| = o(n^{1/2}),$$

$\text{VB}_n f \rightarrow 0$; if (b) the above sums are $O(n^{1/2})$, then $\text{VB}_n f < M$ for all n . Furthermore, (c) there is a bounded skeleton vanishing off some I_A with the sums $O(n^{1/2})$, and such that $\text{VB}_{n_k} f > M$ for some $M > 0$ and a subsequence $\{n_k\}$. Finally, (d) there is a bounded skeleton f which vanishes off some I_A with the above sums of order $O(n^{1/2} A_n)$ for monotone sequences $A_n \rightarrow +\infty$ however slowly, but such that $\text{VB}_n f$ is unbounded.

PROOF. By Remark (2.4)

$$(3.2) \quad \text{VB}_n f \leq 2 \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| \text{Max } p_{nr} = 2 \sum_{r/n \in I_A} \left| f\left(\frac{r}{n}\right) \right| \text{Max } p_{nr}.$$

If we let $x = r/n$, it follows from Remark (2.3) that

$$(3.3) \quad \text{Max } p_{nr} < C n^{1/2} / [r(n-r)]^{1/2}$$

for $r = 1, \dots, n-1$, since for these r , $p_{nr}(0) = p_{nr}(1) = 0$. If $r/n \in I_A$,

then $r > nA$ and $n - r \geq nA$. Hence for $r/n \in I_A$,

$$(3.4) \quad \text{Max } p_{nr} \leq Cn^{1/2}/[nA \cdot nA]^{1/2} = C/A n^{1/2}.$$

Substitution in (3.2) gives

$$(3.5) \quad VB_n f \leq \frac{2C}{A n^{1/2}} \sum_{r/n \in I_A} \left| f\left(\frac{r}{n}\right) \right|.$$

Parts (a) and (b) of the theorem now follow directly from (3.5).

We now prove (d). We can assume $A_n n^{-1/2} \leq 1$. Let $I_A \equiv [1/4, 3/4]$. If $\delta_n \equiv n^{-1/2}$, then for each n , I_A contains $m = [n^{1/2}/4]$ adjacent non-overlapping subintervals $I_k, k = 1, \dots, m$ of length $2\delta_n$. First we define a "partial" skeleton g_n . If $r/n \in I_k$ with k even, we put $g_n(r/n) = 1$ and for all other rationals, put $g_n = 0$. Then if x is the midpoint of an I_k , with k even,

$$(3.6) \quad \begin{aligned} B_n g_n(x) &\geq \sum_{|r/n - x| \leq \delta_n} p_{nr}(x) = 1 - \sum_{|r/n - x| > \delta_n} p_{nr}(x) \\ &\geq 1 - \frac{1}{4n\delta_n^2} = \frac{3}{4} \end{aligned}$$

because of Remarks (2.1) and (2.2). Also, if x' is the midpoint of an I_k , for k odd, then

$$(3.7) \quad B_n g_n(x') \leq \sum_{|r/n - x'| \geq \delta_n} p_{nr}(x') \leq \frac{1}{4}.$$

Hence the oscillation of $B_n g_n(x)$ on $I_k \cup I_{k+1}$ is at least $\frac{1}{2}$, so that

$$(3.8) \quad VB_n g_n \geq \frac{1}{2}(m - 1) \geq \frac{1}{10} n^{1/2}$$

for large n ; also $\sum_{r=0}^n g_n(r/n) < n$. Next we modify the g_n by putting $f_n = A_n n^{-1/2} g_n$. Then $0 \leq f_n \leq 1$ on $[0, 1]$ and

$$(3.9) \quad VB_n f_n \geq \frac{1}{10} A_n, \quad \sum_{r=0}^n f_n\left(\frac{r}{n}\right) \leq A_n n^{1/2}.$$

Now choose $\{n_k\}$ to be a sequence of relatively prime integers increasing so rapidly that

$$(3.10) \quad n_{k-1}^{1/2} A_{n_{k-1}} \leq \frac{1}{2} n_k^{1/2} A_{n_k}.$$

Define a skeleton f by putting $f(r/n_k) = f_{n_k}(r/n_k)$ for $0 \leq r \leq n_k$ and $f = 0$ at other rationals. Then

$$(3.11) \quad VB_{n_k}f = VB_{n_k}f_{n_k} \geq \frac{1}{10} A_{n_k} \rightarrow \infty \quad (n \rightarrow \infty).$$

Also for each n , if n_k denotes the largest n_i with $n_i \leq n$,

$$(3.13) \quad \sum_{r=0}^n f\left(\frac{r}{n}\right) \leq \sum_{i=1}^k \sum_{r=0}^{n_i} f\left(\frac{r}{n_i}\right) \leq \sum_{i=1}^k n_i^{1/2} A_{n_i} \\ \leq 2n_k^{1/2} A_{n_k} \leq 2n^{1/2} A_n.$$

Thus f is bounded, $VB_n f$ unbounded and the sums are $O(n^{1/2} A_n)$. Part (c) follows from the same argument except that we let $f_n \equiv n^{-1/2} g_n$. If we remove the boundedness restriction on f we can simply define f to be $A_{n_k} n_k^{1/2}$ at only one r for each n_k as long as $r/n_k \in I_A$, and 0 elsewhere.

4. An example. Define a skeleton by $f(p/q) = 1/q$ when p and q are relatively prime, and $p/q \in I_A$ for some $0 < A < \frac{1}{2}$. Put $f = 0$ on $[0, 1] - I_A$. By considering n of the form 2^N , it is easy to show $\sum_{r=0}^{n-1} |f((r+1)/n) - f(r/n)|$ is unbounded. Thus f is not of bounded variation even with respect to the rationals, so that neither (1.2) nor Theorem (1.1) applies. However, Theorem (3.1.a) shows that $VB_n f \rightarrow 0$. To see this, let $n = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$ (q_i prime). Now $f(r/n)$ can only assume the values $A_1^{-1} \cdots A_{d(n)-1}^{-1}$ where the $A_i > 1$ are the divisors of n , and $d(n)$ is the number of such divisors. Each value A_i^{-1} is assumed at most $A_i - 1$ times so that

$$(4.1) \quad \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| < \sum_{d(n)-1} 1 < d(n).$$

But $d(n) = o(n^\alpha)$ for any $\alpha > 0$, [1, p. 260] and thus is $o(n^{1/2})$.

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