FREE PRODUCTS OF ORDERED SEMIGROUPS

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Several years ago A. A. Vinogradov [1] proved that the free product of two ordered groups is orderable. In this note we give a simplified version of his proof, which also holds for semigroups.

A semigroup S is said to be ordered iff it has a transitive linear ordering $<$ such that whenever $a < b$ then $ac < bc$ and $ca < cb$ for all $c \in S$. Clearly, each ordered semigroup is cancellative. Since each ordered semigroup can be imbedded in an ordered semigroup with identity (letting $1 < a$ iff $a < a^2$), we can assume without loss of generality that each semigroup considered below has an identity element.

Given two ordered semigroups S_1 and S_2 , we can form their cartesian product $S_1 S_2$ and their free product $S_1 * S_2$. The orders in S_1 and S_2 can easily be extended to $S_1 S_2$, perhaps in many different ways. For example, if $a_i, b_i \in S_i$, we can define $a_1 a_2 > b_1 b_2$ in $S_1 S_2$ iff either $a_1 > b_1$ or $a_1 = b_1$ and $a_2 > b_2$. The aim of this paper is to prove the following result.

THEOREM. *The free product $S_1 * S_2$ of ordered semigroups S_1 and S_2 is orderable. In fact, for each ordering of $S_1 S_2$, there exists an ordering of $S_1 * S_2$ such that the natural homomorphism $S_1 * S_2 \rightarrow S_1 S_2$ is order preserving.*

This theorem is proved by imbedding $S_1 * S_2$ in an ordered semigroup of infinite-dimensional matrices as follows. Let $R = Q(S_1 S_2)$ be the semigroup algebra of $S_1 S_2$ over the rational field Q . For any given order of $S_1 S_2$, we can order R in the familiar way: if $s_i \in S_1 S_2$ and $a_i \in Q$ with $s_1 < s_2 < \dots < s_n$ and all $a_i \neq 0$, let $0 < \sum_{i=1}^n a_i s_i$ iff $0 < a_n$. Clearly, R is an integral domain with unity, and $S_1 S_2 \subset R$.

Consider now the set T of all upper triangular $\omega \times \omega$ matrices over R whose diagonal elements are in $S_1 S_2$. In terms of matrix units $\{e_{ij} \mid i, j = 1, 2, \dots\}$, T consists of all matrices of the form

$$A = \sum_{i,j=1}^{\infty} a_{ij} e_{ij}, \quad a_{ii} \in S_1 S_2, a_{ij} = 0 \text{ if } i > j.$$

Each $A = (a_{ij})$ can be expressed in terms of its diagonals:

$$A = \sum_{k=0}^{\infty} A_k, \quad A_k = \sum_{i=1}^{\infty} a_{i+k} e_{i+k}.$$

Received by the editors May 17, 1966 and, in revised form, March 2, 1967.

If $B = \sum B_k \in T$, then the k th diagonal of AB is

$$(AB)_k = \sum_{i=0}^k A_i B_{k-i}.$$

It is evident that T is a multiplicative semigroup with identity I .

The semigroup T can be ordered relative to diagonals. In the first place, if $A_k = \sum a_{i \ i+k} e_{i \ i+k}$ and $B_k = \sum b_{i \ i+k} e_{i \ i+k}$ are k th diagonal matrices, let $A_k < B_k$ iff there exists an integer r such that $a_{i \ i+k} = b_{i \ i+k}$ for all $i < r$ and $a_{r \ r+k} < b_{r \ r+k}$. Then for any $A = \sum A_k$ and $B = \sum B_k$ in T , let $A < B$ iff there exists an integer k such that $A_i = B_i$ for all $i < k$ and $A_k < B_k$. It is easily shown that $<$ is a transitive linear ordering in T .

To complete the proof that T is an ordered semigroup, let $A = \sum A_k$, $B = \sum B_k$, and $C = \sum C_k$ be in T , with $A < B$. Hence, $A_i = B_i$ if $i < k$ and $A_k < B_k$ for some integer k . Since $(AC)_i = (BC)_i$ and $A_i C_{k-i} = B_i C_{k-i}$ if $i < k$, and $A_k C_0 < B_k C_0$, evidently $(AC)_k < (BC)_k$. Therefore, $AC < BC$. Similarly, we can show $CA < CB$. Hence, T is an ordered semigroup.

We imbed $S_1 S_2$ in T by identifying each $x \in S_1 S_2$ with the diagonal matrix

$$D_x = (x - 1)D_0 + I = \text{diag}(1, x, 1, x, \dots),$$

where

$$D_0 = \sum_{i=1}^{\infty} e_{2i \ 2i}.$$

Clearly the order in T is an extension of the order in $S_1 S_2$.

The injection of $S_1 * S_2$ in T is accomplished as follows. Let the diagonal matrices A_1 and B_1 be defined by

$$A_1 = \sum_{i=1}^{\infty} e_{2i-1 \ 2i}, \quad B_1 = \sum_{i=1}^{\infty} e_{2i \ 2i+1}.$$

Then $A_1^2 = B_1^2 = 0$, whereas $A_1 B_1$ and $B_1 A_1$ are nonnilpotent diagonal matrices. Also, $A_1 D_0 = A_1$, $D_0 A_1 = 0$, $B_1 D_0 = 0$, and $D_0 B_1 = B_1$. For each $x \in S_1 S_2$, let us define

$$\begin{aligned} H_x &= (I + A_1)^{-1} D_x (I + A_1) = D_x + (1 - x) A_1, \\ K_x &= (I - B_1)^{-1} D_x (I - B_1) = D_x + (1 - x) B_1. \end{aligned}$$

Now we define the mapping $S_1 * S_2 \rightarrow T$ as follows:

$$\phi(\dots xy \dots) = \dots H_x K_y \dots, \quad x \in S_1, y \in S_2.$$

Since $\phi(x) = H_x$ and $\phi(y) = K_y$ for $x \in S_1$, $y \in S_2$, evidently $\phi|_{S_1}$ and $\phi|_{S_2}$ are homomorphisms. Hence, ϕ itself is a homomorphism.

To prove that ϕ is actually a monomorphism, let $x = a_1 a_2 \cdots a_n$ be an element of $S_1 * S_2$ of length n . Thus, $a_i \in S_1$ if i is odd, $a_i \in S_2$ if i is even (or vice versa), and all $a_i \neq 1$. The last nonzero diagonal of $\phi(x) = H_{a_1} K_{a_2} \cdots$ is the n th diagonal as given below:

$$\prod_{i=1}^n (1 - a_i) \sum_{j=1}^{\infty} e_{2j-1 \ 2j+n-1}.$$

If $y = b_1 \cdots b_m$ is an element of $S_1 * S_2$ of length m , then the m th diagonal of $\phi(y)$ is

$$\prod_{i=1}^m (1 - b_i) \sum_{j=1}^{\infty} e_{2j-1 \ 2j+m-1}$$

or

$$\prod_{i=1}^m (1 - b_i) \sum_{j=1}^{\infty} e_{2j \ 2j+m},$$

depending on whether $b_1 \in S_1$ or $b_1 \in S_2$. Before $\phi(x)$ can equal $\phi(y)$, evidently m must equal n and $a_1, b_1 \in S_1$ (or $a_1, b_1 \in S_2$).

Let us assume that for all elements of $S_1 * S_2$ of length less than n , whenever $\phi(x) = \phi(y)$ then $x = y$. Assume that $x = a_1 \cdots a_n$, $y = b_1 \cdots b_n$ have length n , that $a_1, b_1 \in S_1$, and that $\phi(x) = \phi(y)$. Since the n th diagonals of $\phi(x)$ and $\phi(y)$ are equal,

$$(1) \quad \prod_{i=1}^n (1 - a_i) = \prod_{i=1}^n (1 - b_i).$$

The $(n-1)$ th diagonal of $\phi(x)$ is

$$D_{a_1}(1 - a_2)B_1(1 - a_3)A_1 \cdots + (1 - a_1)A_1(1 - a_2)B_1 \cdots D_{a_n}.$$

Thus, the even-row coordinates of the $(n-1)$ th diagonal of $\phi(x)$ are all equal to $a_1(1 - a_2) \cdots (1 - a_n)$. Since $\phi(x) = \phi(y)$, we must have

$$(2) \quad a_1 \prod_{i=2}^n (1 - a_i) = b_1 \prod_{i=2}^n (1 - b_i).$$

From (1) and (2) and the fact that R is an integral domain, we get $a_1 = b_1$. If $x' = a_2 \cdots a_n$ and $y' = b_2 \cdots b_n$, then $\phi(a_1)\phi(x') = \phi(b_1)\phi(y')$, and hence $\phi(x') = \phi(y')$. Therefore, by the induction assumption, $x' = y'$ and $x = y$. An analogous argument holds if $a_1, b_1 \in S_2$. We conclude that ϕ is a monomorphism. Therefore, the ordering in T induces an ordering in $S_1 * S_2$.

If $x \in S_1 * S_2$ maps into $x' \in S_1 S_2$ under the natural homomorphism $S_1 * S_2 \rightarrow S_1 S_2$, then $\phi(x)$ has main diagonal $D_{x'}$. Since the ordering in T is an extension of the ordering in $S_1 S_2$, evidently the natural homomorphism $S_1 * S_2 \rightarrow S_1 S_2$ is order preserving. This completes the proof of the theorem.

Once we have shown how to order the free product of two ordered semigroups, we can order the free product of any well-ordered set of ordered semigroups by induction.

A feature of the construction in the proof above is that it allows us to compute the relationship between elements in any specific case. As a simple example, let $S_1 = \{x_1^a \mid a \in Q^+\}$ and $S_2 = \{x_2^a \mid a \in Q\}$, where x_1 and x_2 are indeterminates, Q is the additive group of rationals, and $Q^+ = \{a \in Q \mid a \geq 0\}$. We can order $S_1 S_2$ by total degrees as follows: $x_1^{a_1} x_2^{a_2} > x_1^{b_1} x_2^{b_2}$ iff $a_1 + a_2 > b_1 + b_2$ or $a_1 + a_2 = b_1 + b_2$ and $a_1 > b_1$. If, for example, $y = x_2^{-1} x_1 x_2$, then $\phi(y) = D_{x_1} + (x_2 - x_1 x_2) A_1 + (1 - x_1 - x_2 + x_1 x_2) B_1 + \text{higher diagonals}$. Since $x_2 - x_1 x_2 < 0$, we have $y < x_1$. Similarly, if $z = x_2 x_1 x_2^{-1}$ then $\phi(z) = D_{x_1} + (x_2^{-1} - x_1 x_2^{-1}) A_1 + (1 - x_1 - x_2^{-1} - x_1 x_2^{-1}) B_1 + \text{higher diagonals}$ and $z < x_1$. Also, $x_2^{-1} - x_1 x_2^{-1} > x_2 - x_1 x_2$ and therefore $z > y$.

REFERENCE

1. A. A. Vinogradov, *On the free product of ordered groups*, Mat. Sb. 25 (1949), 163-168. (Russian)

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