ON THE UNIFORM STABILITY OF A PERTURBED LINEAR FUNCTIONAL DIFFERENTIAL EQUATION

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The objective of this note is to extend to perturbed linear functional differential equations previous results [4] concerning the uniform stability of a perturbed linear ordinary differential system. This is done in the Theorem and the Corollary stated below. The Corollary is also a generalization of a result of Halanay [1, Theorem 4.7] on uniform stability. The main tool used in [4] is the Tychonov fixed point theorem. In this work, the basic tool is provided by an interesting result of J. Hale [2, Theorem II-3], in which the problem of determining the behavior of the solutions of a functional differential equation is reduced to the solution of an ordinary scalar differential equation. Hale's result is based on use of Lyapunov functionals. An extension of results in [4], for the case of a perturbed nonlinear ordinary differential equation, was done by Strauss [5].

Let C be the space of continuous functions taking the interval $[-h, 0]$, $0 \leqq h < \infty$, into the space R^n of n-dimensional real vectors. Here $|\cdot|$ will denote any convenient vector norm in R^n and for any $\phi \in C$, $\|\phi\| = \sup_{-h \leqq \theta \leqq 0} |\phi(\theta)|$. If x is a continuous function on $[-h+t_0, t_0+A)$, $0 < A \leqq \infty$, x_t is defined by $x_t(\theta) = x(t+\theta)$, $-h \leqq \theta \leqq 0$, $t_0 \leqq t < t_0 + A$. For H, $0 < H \leqq \infty$, $C_H = \{\phi \in C \mid \|\phi\| < H\}$.

Let $f(\phi, t) \in R^n$ be a function defined on $C_H \times [0, \infty)$. Then

$$\dot{x}(t) = f(x_t, t) \tag{1}$$

where $(\cdot)$ denotes right-hand derivative, is called a functional differential equation. For the special case in which $h = 0$, (1) is an ordinary differential system.

Let t_0 be any nonnegative number and $\phi \in C_H$ be any given function. A continuous function $x(t)$ defined on $[t_0 - h, t_0 + A)$, $A > 0$, is said to be a solution of (1) with initial function ϕ at $t = t_0$ if:

(i) for each t, $t_0 \leqq t < t_0 + A$, $x_t \in C_H$,

(ii) $x_{t_0} = \phi$,

(iii) $\dot{x}(t) = f(x_t, t)$, for $t_0 \leqq t < t_0 + A$.

Function $f(\phi, t)$ is said to be locally Lipschitzian in $C_H \times [0, \infty)$ if for every H_1, $0 < H_1 < H$, and every $T > 0$, there is $L = L(T, H_1)$, such that $|f(\phi, t) - f(\psi, t)| \leqq L\|\phi - \psi\|$, for $\phi, \psi \in C_{H_1}$, $0 \leqq t \leqq T$.

In this note $f(\phi, t)$ is supposed to be continuous and locally Lipschitzian in $C_H \times [0, \infty)$.

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The unique solution of (1) with initial function ϕ at $t=t_0$ will be denoted by $x(t)=x(t, t_0, \phi)$ and $x_t=x_t(t_0, \phi)$; $[t_0-h, t^+)$ will denote the maximal interval, open at the right, in which $x(t)$ is defined. If $t^+=\infty$, the solution is said to be defined in the future and if in addition $x(t)$ is bounded on $[t_0, \infty)$, the solution is said to be bounded in the future. If $x_t \in C_{H_1}$ for $t_0 \leqq t < t^+$, $H_1 < H$, then $t^+=\infty$.

Concerning (1), the following lemma holds:

LEMMA 1. *Suppose $t_0 \geqq 0$, ϕ, $\psi \in C_H$, and $x(t, t_0, \phi)$, $x(t, t_0, \psi)$ defined on a common interval $[t_0-h, t_0+A]$, $A>0$. Then*

$$|x(t, t_0, \phi) - x(t, t_0, \psi)| \leqq [\exp L(t - t_0)]\|\phi - \psi\|, \quad t_0 \leqq t < t_0 + A.$$

For a proof of this lemma, see N. Krasovskii [**3**, p. 128].

A solution $x(t, t_0, \phi)$ of (1), defined in the future, is said to be stable if for every $\epsilon>0$ and every $t_1 \geqq t_0$, there exists a $\delta=\delta(t_1, \epsilon)>0$ such that, if $\|x_{t_1}(t_0, \phi)-\psi\|<\delta$, then $x_t(t_1, \psi) \in C_H$ and $\|x_t(t_0, \phi)-x_t(t_1, \psi)\|<\epsilon$ for all $t \geqq t_1$. If $\delta(t_1, \epsilon)$ can be chosen independent of t_1, then $x(t, t_0, \phi)$ is said to be uniformly stable. As a consequence of Lemma 1 one can see that, if for every $\epsilon>0$ there exists $T(\epsilon) \geqq t_0$ such that the above condition in the definition of stability (uniform stability) holds for every $t_1 \geqq T(\epsilon)$, then the solution is stable (uniformly stable). For ordinary differential equations if the above condition in the definition of stability holds for some $\bar{t}_0$, then it holds for every $t_0 \geqq 0$. This is a consequence of the fact that, for ordinary differential equations, the map $x_t(t_0, \phi)$ taking the initial value ϕ into the value of the solution at the time t induces an homeomorphism between spheres. But this is no longer true for the general case of functional differential equations. See, for instance, an example of Zverkin [**5**], commented on in [**1**, p. 6].

Consider the equation

$$\dot{y}(t) = L(y_t, t) \tag{2}$$

where $L(\phi, t)$ is continuous on $C \times [0, \infty)$ and linear in ϕ. One can see that if the zero solution is stable (uniformly stable), then (2) is stable (uniformly stable), that is, every solution $y(t, t_0, \phi)$ is stable (uniformly stable). Concerning uniform stability of linear systems, the following lemma holds:

LEMMA 2. *Equation (2) is uniformly stable if, and only if, there is a positive constant K, such that $|y_t(t_0, \phi)| \leqq K\|\phi\|$ for every $t \geqq t_0 \geqq 0$ and $\phi \in C$.*

PROOF. Suppose that the linear system is uniformly stable. Then there exists a $\delta>0$, such that $|y(t, t_0, \psi)| \leqq 1$ for every $t \geqq t_0$ and

$\|\psi\| \leqq \delta$. Then for every $\phi \in C$, $\phi \neq 0$, and $t \geqq t_0 \geqq 0$, it follows that $|y(t, t_0, \delta\phi/\|\phi\|)| \leqq 1$, that is, $|y(t, t_0, \phi)| \leqq K\|\phi\|$, with $K = 1/\delta$.

As the converse is obvious, the proof is complete.

Consider the system

$$\dot{x}(t) = L(x_t, t) + X(x_t, t) \tag{3}$$

where $L(\phi, t)$ is continuous and linear in ϕ and $X(\phi, t)$ is continuous and locally Lipschitzian in $C \times [0, \infty)$.

The following lemma, which plays an important role in proving the Theorem given below, is a particular case of the above mentioned Hale's result [2, Theorem II-3].

Lemma 3. *Suppose that the following conditions are satisfied:*

(i) *There exists a positive constant K, such that for every $t_0 \geqq 0$ and every $t \geqq t_0$, $\phi \in C$, the solution $y(t, t_0, \phi)$ of* (2) *satisfies* $|y(t, t_0, \phi)| \leqq K\|\phi\|$;

(ii) *There exists a continuous function $\omega(r, t)$ defined for $t \geqq 0$, $0 \leqq r < \infty$, nondecreasing in r, such that*

$$|X(\psi, t)| \leqq \omega(\|\psi\|, t), \quad t \geqq 0, \quad \psi \in C_H, \quad 0 < H < \infty;$$

(iii) *The maximal solution $u(t)$ of the ordinary scalar differential equation $\dot{u} = K\omega(u, t)$, $u(t_0) = \sup_{s \geqq 0}\|y_{t_0+s}(t_0, \phi)\|$, with $\phi \in C_H$, satisfies $u(t) < H$ for $t \geqq t_0$.*

Then the solution $x(t, t_0, \phi)$ of (3) *satisfies the relation*

$$\|x_t(t_0, \phi)\| \leqq u(t), \quad t_0 \leqq t < \infty.$$

As a consequence of Lemma 3, we have the following lemma.

Lemma 4. *Suppose that condition* (i) *of Lemma* 3 *holds and that*

(iv) *for every positive H there exists a function $h_H(t)$, $0 \leqq t < \infty$, with $\int^\infty h_H(t)dt < \infty$, such that $|X(\psi, t)| \leqq h_H(t)$, for all (t, ψ), $0 \leqq t < \infty$, $\|\psi\| \leqq H$.*

Let $\phi \in C$, $t_0 \geqq 0$, $0 < H < \infty$ be given such that

$$K\left[\|\phi\| + \int_{t_0}^\infty h_H(s)ds\right] < H.$$

Then the solution $x(t, t_0, \phi)$ of (3) *satisfies*

$$\|x_t(t_0, \phi)\| \leq K\left[\|\phi\| + \int_{t_0}^t h_H(s)ds\right], \quad t \geqq t_0.$$

Proof. Take $\omega(r, t) = h_H(t)$. The maximal solution of the ordinary scalar differential equation

$$\dot{u}(t) = Kh_H(t), \quad u(t_0) = \sup_{s \geqq 0} | y_{t_0+s}(t_0, \phi) |$$

is $u(t) = u(t_0) + \int_{t_0}^{t} Kh_H(\tau)d\tau$ and, by using condition (i), it follows that

$$u(t) \leqq K\left[||\phi|| + \int_{t_0}^{t} h_H(\tau)d\tau\right] \leqq K\left[||\phi|| + \int_{t_0}^{\infty} h_H(\tau)d\tau\right] < H.$$

Then, by applying Lemma 3, the following relation holds:

$$| x(t, t_0, \phi) | \leqq K\left[||\phi|| + \int_{t_0}^{t} h_H(\tau)d\tau\right], \qquad t \geqq t_0 .$$

The proof is complete.

We proceed now to the main result of this note.

THEOREM. *Suppose that* (2) *is uniformly stable and that condition* (iv) *of Lemma* 4 *holds.*

Then every solution of (3), *bounded in the future, is uniformly stable.*

REMARK. There are indeed solutions of (3) bounded in the future. In fact, let $\phi \in C$ and $t_0 \geqq 0$ satisfying $||\phi|| < H/2K$ and $\int_{t_0}^{\infty} h_H(s)ds < H/2K$, where $H > 0$ and K is given in Lemma 2. Then for such ϕ and t_0, one obtains $||x_t(t_0, \phi)|| < H$, using Lemma 4.

PROOF OF THEOREM. Let $x(t, t_0, \phi)$ be a solution of (3), bounded in the future. By making the change of variable $z = x - x(t, t_0, \phi)$ in (3), one can see that the new equation is given by

$$\dot{z}(t) = L(z_t, t) + Y(z_t, t)$$

where $Y(\psi, t) = X(\psi + x_t(t_0, \phi), t) - X(x_t(t_0, \phi), t)$ satisfies hypothesis (iv) and $Y(0, t) = 0$.

This shows that there is no loss of generality in supposing $X(0, t) = 0$ and it is enough to prove that the solution $x = 0$ of (3) is uniformly stable.

Given a positive ϵ, there exists $T = T(\epsilon)$, $\delta = \delta(\epsilon)$, such that the following inequality holds for every $t_1 \geqq T$ and $||\phi|| < \delta$:

$$K\left[||\phi|| + \int_{t_1}^{\infty} h_\epsilon(s)ds\right] < \epsilon.$$

As (2) is uniformly stable, it follows from Lemma 2 that the hypotheses of Lemma 4 are satisfied. Then the solution $x(t, t_1, \phi)$ of (3), where $t_1 \geqq T(\epsilon)$ and $||\phi|| < \delta$, satisfies the relation

$$| x(t, t_1, \phi) | \leqq K\left[||\phi|| + \int_{t_1}^{t} h_\epsilon(s)ds\right] < \epsilon, \qquad t \geqq t_1.$$

By the remark following the definition of uniform stability, the above relation implies that the solution $x=0$ of (3) is uniformly stable.

COROLLARY. *Suppose that* (2) *is uniformly stable and that the following relation holds*:

$$|X(\phi, t)| \leqq h(t)\|\phi\|, \quad \textit{with} \int_0^\infty h(t)dt < \infty.$$

Then every solution of (3) *is bounded in the future and uniformly stable.*

PROOF. By using Gronwall's inequality it is not hard to see that every solution of (3) is defined in the future. Let us show now that every solution of (3) is bounded in the future. As every solution is defined in the future, it is enough to prove that there is a $T \geqq 0$ such that for every $t_0 \geqq T$ and every $\phi \in C$, $x(t, t_0, \phi)$ is bounded in the future.

Let us take $T \geqq 0$ such that $K\int_T^\infty h(t)dt < 1$. Let $t_0 \geqq T$ and $\phi \in C$. We can choose H large enough to satisfy the relation

$$K\left[\|\phi\| + \int_T^\infty Hh(t)dt\right] < H.$$

By taking in Lemma 4, $h_H(t) = Hh(t)$, it follows that the solution $x(t, t_0, \phi)$ of (3) satisfies the relation $\|x_t(t_0, \phi)\| < H$.

As every solution of (3) is bounded in the future, then the Corollary follows from the Theorem. The proof is complete.

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