# A COHOMOLOGICAL CHARACTERIZATION OF FINITE NILPOTENT GROUPS[1]

W. J. WONG

In proving the existence of outer automorphisms of finite $p$-groups [2], Gaschütz showed that if $A$ is a finite module with $p$-power order over a finite $p$-group $G$ such that $H^1(G, A)=0$, then $H^n(H, A)=0$ for all $n \geqq 1$ and all $H \leqq G$. In this note (whose origin lies in some remarks made by Professor Hans Zassenhaus)[2] we use Gaschütz's method to prove a similar result valid for all finite nilpotent groups and show that this result fails for every finite nonnilpotent group so that we have a characterization of the class of finite nilpotent groups.

We let $H^n(G, A)$ denote the $n$th cohomology group of $G$ in $A$ in the sense of Tate (cf. [6]). We recall the method of dimension-shifting. If $A$ is any $G$-module and $Z(G)$ is the group ring of $G$ over the ring of integers, then $Z(G) \otimes A$ is a $G$-module in a natural way, and there exist exact sequences of $G$-modules

$$0 \rightarrow A \rightarrow Z(G) \otimes A \rightarrow B \rightarrow 0,$$

$$0 \rightarrow C \rightarrow Z(G) \otimes A \rightarrow A \rightarrow 0.$$

Now, $Z(G) \otimes A$ is $G$-regular so that $H^n(G, Z(G) \otimes A)=0$ for all $n$, and we have

$$H^n(G, A) \cong H^{n-1}(G, B) \cong H^{n+1}(G, C)$$

for all $n$. When $A$ is finite then $Z(G) \otimes A$ is finite, so that $B$ and $C$ are also finite.

THEOREM 1. *Let $G$ be a finite nilpotent group and $A$ a finite $G$-module such that $H^m(G, A)=0$ for some $m$. Then $H^n(G, A)=0$ for all $n$.*

PROOF. If $A_p$ is the Sylow $p$-subgroup of $A$, then $H^n(G, A) = \sum H^n(G, A_p)$, where the direct sum is taken over all primes $p$. It follows that we may assume that $A$ is a $p$-group for some prime $p$. By dimension-shifting, we may assume that $m=1$.

The result is valid if $p$ does not divide $|G|$; in particular, it is valid when $|G|=1$. Thus we use induction on $|G|$ and may assume that $p$ divides $|G|$. Since $G$ is nilpotent, we can find a normal subgroup $H$ of index $p$ in $G$. We have the exact sequence [3, p. 129]:

Received by the editors March 1, 1967.

[1] Research supported by NSF grant GP-3986 at the University of Notre Dame.

[2] Professor Zassenhaus has since informed me that he, K. Hoechsmann and P. Roquette have also obtained the results of this note.

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H} \to H^2(G/H, A^H).$$

Since $H^1(G, A)=0$ by hypothesis, we have $H^1(G/H, A^H)=0$. Since $G/H$ is cyclic and the Herbrand quotient of the finite module $A^H$ is 1 [1, p. 57], we have

$$H^n(G/H, A^H) = 0, \quad \text{all } n. \tag{1}$$

It follows that $H^1(H, A)^{G/H}=0$. Since $G/H$ and $H^1(H, A)$ are $p$-groups, this implies that $H^1(H, A)=0$. By induction hypothesis, we have

$$H^n(H, A) = 0, \quad \text{all } n. \tag{2}$$

Now we have the inflation-restriction and transfer-deflation exact sequences [3, p. 129], [6, p. 321]:

$$H^n(G/H, A^H) \to H^n(G, A) \to H^n(H, A), \qquad n \geqq 1,$$
$$H^n(H, A) \to H^n(G, A) \to H^n(G/H, A^H), \qquad n \leqq 0.$$

From (1) and (2) it now follows that $H^n(G, A)=0$, all $n$.

THEOREM 2. *Let $G$ be a finite nonnilpotent group, $n$ any integer. Then there exists a finite $G$-module $A$ such that $H^n(G, A)\neq 0$, $H^{n+1}(G, A)=0$.*

PROOF. By dimension-shifting, we may assume that $n=0$. Let $H$ be a minimal nonnilpotent subgroup of $G$. By a theorem of Schmidt and Golfand [4, p. 304], the commutator subgroup of $H$ is a Sylow $p$-subgroup of $H$ for some prime $p$. Let $B$ be the trivial $H$-module of order $p$. Then,

$$H^0(H, B) \cong B \neq 0, \qquad H^1(H, B) \cong \mathrm{Hom}(H, B) = 0.$$

Now let $A$ be the $G$-module induced by $B$ (denoted $M_G^H(B)$ in [5, p. 1–12]). Then $H^n(G, A)\cong H^n(H, B)$ for all $n$, so that

$$H^0(G, A) \neq 0, \qquad H^1(G, A) = 0.$$

Moreover, $A$ is finite since $B$ is finite.

REMARK. If $G$ is a finite $p$-group and $A$ a finite $G$-module such that $H^m(G, A)=0$ for some $m$, then $H^n(H, A)=0$ for all $n$ and all subgroups $H$ of $G$, by the proof of Theorem 1, since every proper subgroup of $G$ is contained in one of index $p$. However, this statement is false whenever $G$ is not a $p$-group. For, suppose that $G$ is a finite nilpotent group whose order is divisible by at least two primes. Let $p$ be one of these prime divisors of $|G|$ and let $H$ be a Sylow $p$-subgroup of $G$. Since $H$ is normal in $G$ and $G/H$ is a nontrivial $p'$-group, there exists a nontrivial irreducible $K(G)$-module $A$ on which $H$ acts trivially, where $K$ is a finite field of characteristic $p$ and $K(G)$ is the group algebra of $G$ over $K$. Then $H^0(G, A)=0$ but $H^0(H, A)\cong A\neq 0$. Theorem 1 shows that in fact $H^n(G, A)=0$ and $H^n(H, A)\neq 0$, all $n$.

## References

1. C. Chevalley, *Class field theory*, Nagoya University, Nagoya, 1954.

2. W. Gaschütz, *Kohomologische Trivialitäten und äussere Automorphismen von p-Gruppen*, Math. Z. **88** (1965), 432–433.

3. G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–134.

4. L. Rédei, *Die endlichen einstufig nichtnilpotenten Gruppen*, Publ. Math. Debrecen **4** (1956), 303–324.

5. J.-P. Serre, *Cohomologie galoisienne*, Springer, Berlin, 1965.

6. E. Weiss, *A deflation map*, J. Math. Mech. **8** (1959), 309–329.

University of Notre Dame