ISOMORPHISMS OF PICKERT-MOULTON PLANES

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Pickert-Moulton Planes were defined more than a decade ago [2, p. 93], but the condition that two such planes be isomorphic has not appeared in the literature. Here, I shall give a simple criterion for the existence of such an isomorphism, and determine explicitly the most general isomorphism.

A Pickert-Moulton Plane over an ordered field F has points (x, y) , and lines $[x=c]$, $[y=c+r \circ x]$; where

$$\begin{aligned} r \circ x &= prx && \text{for } r, x < 0, \\ &= rx && \text{otherwise;} \end{aligned}$$

$p > 0$, c, r, x, y being in F . (In F. R. Moulton's classic model [1], the constant p is 2 and F is the field of real numbers.)

The (generalized) Moulton Planes which I defined in [3], have

$$\begin{aligned} r \circ x &= rx && \text{for all } x \geq 0 \\ &= \phi(r) \cdot x && \text{for all } x < 0 \end{aligned} \quad \text{and all } r \text{ (in } F\text{)}.$$

Over an *ordered* field F , ϕ is an increasing function of F onto F (normalized by $\phi(0)=0$, $\phi(1)=1$), restricted only by the condition: $x \rightarrow \phi(x) + x$ maps F onto F [3, p. 429, Theorem 1]. The Pickert-Moulton Plane is a special "Moulton Plane" with F ordered, and

$$\begin{aligned} \phi(r) &= r && \text{for } r \geq 0, \\ &= pr && \text{for } r < 0. \end{aligned}$$

Thus the collineation theorems of [4] and [5] apply, in particular, to Pickert-Moulton Planes.

J. C. D. Spencer (Mrs. Yaqub) [6] has given explicit formulae for certain collineations on a Pickert-Moulton Plane. In [4] and [5], I determined the full collineation group for such a plane—in fact, for any Moulton Plane. Also, I characterized a Pickert-Moulton Plane among *all* Moulton Planes in terms of its collineation group; viz., as one supporting a collineation (onto itself) that displaces the ideal point on $[x=0]$. (Cf. [5, p. 650, Theorem 3].)

THEOREM 1. *Let $M_\phi(F)$ and $M_\psi(K)$ be Pickert-Moulton Planes over the respective ordered fields F and K , where*

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$$\begin{aligned}\phi(r) &= r & \text{for } r \text{ (in } F) \geq 0, & & \psi(r) &= r & \text{for } r \text{ (in } K) \geq 0, \\ &= pr & \text{for } r \text{ (in } F) < 0; & & &= qr & \text{for } r \text{ (in } K) < 0.\end{aligned}$$

Then $M_\phi(F)$ and $M_\psi(K)$ are isomorphic if and only if there exists an order-preserving isomorphism α of F onto K such that $q = \alpha(p)$ or $q \cdot \alpha(p) = 1$.

Proof. The general form of the collineation in [5, p. 650, Theorem 3] shows that $M_\phi(F)$ is isomorphic to $M_\psi(K)$ if and only if there is an affine collineation of $M_\phi(F)$ onto $M_\psi(K)$. This, by [4, p. 54, Theorem 1], is equivalent to the existence of an order-preserving isomorphism α , from F onto K , and constants e , $a \neq 0$ in K , such that either

$$\begin{aligned}(1) \quad & \text{(Case (i))} \quad \alpha\phi(r) = \frac{\psi[a \cdot \alpha(r) + e] - \psi(e)}{\psi(a + e) - \psi(e)}, \quad \text{or} \\ & \text{(Case (ii))} \quad \alpha(r) = \frac{\psi[a \cdot \alpha\phi(r) + e] - \psi(e)}{\psi(a + e) - \psi(e)}\end{aligned}$$

for all r in F .

Formulae (1) can be combined by putting

$$(2) \quad \begin{aligned}u' &= \phi(r) & \text{and} & & u &= r & \text{for Case (i),} \\ &= r & & & &= \phi(r) & \text{for Case (ii),}\end{aligned} \quad \text{all } r \text{ in } F.$$

The composite formula is

$$(3) \quad \alpha(u') = \frac{\psi[a \cdot \alpha(u) + e] - \psi(e)}{\psi(a + e) - \psi(e)},$$

where

$$(4) \quad u' = u, \quad \text{for all } u \geq 0, \quad = ku \quad \text{for all } u < 0;$$

k being p in Case (i), but p^{-1} in Case (ii). By definition,

$$\begin{aligned}\psi[a \cdot \alpha(u) + e] &= a \cdot \alpha(u) + e & \text{for } a \cdot \alpha(u) + e \geq 0, \\ &= qa \cdot \alpha(u) + qe & \text{for } a \cdot \alpha(u) + e \leq 0,\end{aligned} \quad \text{all } u \text{ in } F.$$

Substituting this in (3) gives

$$\begin{aligned}(5) \quad u' &= \frac{\alpha^{-1}(a) \cdot u + \alpha^{-1}[e - \psi(e)]}{\alpha^{-1}[\psi(a + e) - \psi(e)]} & \text{as } \{[\alpha^{-1}(a)] \cdot u + \alpha^{-1}(e)\} \geq 0, \\ &= \frac{\alpha^{-1}(qa) \cdot u + \alpha^{-1}[qe - \psi(e)]}{\alpha^{-1}[\psi(a + e) - \psi(e)]} & \text{as } \{[\alpha^{-1}(a)] \cdot u + \alpha^{-1}(e)\} \leq 0,\end{aligned}$$

for all u in F . Comparison of (4) and (5) shows that $e = \psi(e) = 0$, and that the (unordered) pair

$$(6) \quad \begin{aligned} \{\alpha^{-1}[a/\psi(a)], \alpha^{-1}[qa/\psi(a)]\} &= [1, k] = [1, p] \text{ in Case (i),} \\ &= [1, p^{-1}] \text{ in Case (ii).} \end{aligned}$$

Thus, either $a/\psi(a) = \alpha(1) = 1$ and $qa/\psi(a) = \alpha(k)$; or $a/\psi(a) = \alpha(k)$ and $qa/\psi(a) = \alpha(1) = 1$. In either case, $q = \alpha(p)$ or $q \cdot \alpha(p) = 1$.

We conclude that the required (order-preserving isomorphism) α exists if and only if $q = \alpha(p)$ or $q \cdot \alpha(p) = 1$, and the theorem follows.

COROLLARY. *If α is the identity (as it must be when $F = K =$ the real field), the condition for isomorphism reduces to $q = p$ or $q = p^{-1}$.*

The isomorphisms between any two Moulton Planes were completely determined in [4] and [5]. It should be of interest, however, to exhibit the formulae for the special case of the Pickert-Moulton Planes; and this is done in Theorem 2 below. The notation of Theorem 1 will be retained.

THEOREM 2. (A) *The most general affine isomorphism β of $M_\phi(F)$ onto $M_\psi(K)$ sends (x, y) to (x', y') ; where $y' = g \cdot \alpha(y) + d$ for all y in F , and*

$$\begin{aligned} x' = \mu(x) &= g/a \cdot \alpha(x) && \text{for } x' \geq 0, \\ &= g/\psi(a) \cdot \alpha(x) && \text{for } x' < 0, \end{aligned}$$

all x and nonzero constants g, a in F .

For any r in F , $\beta[P_\infty(r)] = P_\infty(r')$, where

$$\begin{aligned} \alpha(r) &= r'/a && \text{in Case (i) } \mu(1) > 0, \\ &= \psi(r')/\psi(a) && \text{in Case (ii) } \mu(1) < 0; \end{aligned}$$

$\beta(Y_\infty) = Y_\infty^c$.

(B) *The most general nonaffine isomorphism γ of $M_\phi(F)$ onto $M_\psi(K)$ is $\sigma\beta$; with β the general affine isomorphism (Part A), and σ a special collineation on $M_\phi(F)$. Let k be an arbitrary constant in F . For all x and all $y \neq (-1 - k)$ in F , $\sigma[(x, y)] = (x', y')$ where*

$$\begin{aligned} y' &= \frac{y + k}{y + k + 1}, \\ x' &= \frac{x}{p \cdot (y + k + 1)} && \text{for } x \geq 0 \text{ and } y + 1 + k < 0, \\ &= \frac{x}{y + k + 1} && \text{otherwise.} \end{aligned}$$

The map σ is completed as follows: $\sigma[(x, -1-k)] = P_\infty(-1/x)$ for all $x \neq 0$; $\sigma[(0, -1-k)] = Y_\infty$; $\sigma(Y_\infty) = (0, 1)$; $\sigma[P_\infty(0)] = P_\infty(0)$; and

$$\begin{aligned}\sigma[P_\infty(r)] &= (1/r, 1) && \text{for } r > 0, \\ &= (1/pr, 1) && \text{for } r < 0,\end{aligned}$$

(C) The general nonaffine collineation γ may be summarized explicitly as follows. If $y \neq -1-k$, and x is in F , then $\gamma[(x, y)] = (x', y')$; where

$$y' = g \cdot \alpha[(y+k)/(y+k+1)] + d, \quad \text{for all } y \text{ in } F;$$

the line $y = -1-k$ in $M_\phi(F)$ maps onto the ideal line of $M_\psi(K)$; and the ideal line of $M_\phi(F)$ maps onto $\{y' = g+d\}$ in $M_\psi(K)$; $\gamma[P_\infty(0)] = P_\infty(0)$; $\gamma(Y_\infty) = (0, g+d)$; and $\gamma[(0, -1-k)] = Y_\infty$.

Case (i). $\mu(1) = g/a > 0$.

For $x \neq 0$, $\gamma[(x, -1-k)] = P_\infty[a \cdot \alpha(-1/x)]$;

$$\begin{aligned}\gamma[P_\infty(r)] &= (g/[a \cdot \alpha(r)], g+d) && \text{for } r > 0, \\ &= (g/[\psi(a) \cdot \alpha(pr)], g+d) && \text{for } r < 0;\end{aligned}$$

and

$$\begin{aligned}x' &= [g/\psi(a)] \cdot \alpha\{x/[p \cdot (y+k+1)]\} && \text{for } x \geq 0 \text{ and } (y+k+1) < 0, \\ &= (g/a) \cdot \alpha[x/(y+k+1)] && \text{for } x \cdot (y+k+1) > 0, \\ &= [g/\psi(a)] \cdot \alpha[x/(y+k+1)] && \text{for } x \leq 0 \text{ and } (y+k+1) > 0.\end{aligned}$$

Case (ii). $\mu(1) = g/\psi(a) < 0$.

For $x \neq 0$, $\gamma[(x, -1-k)] = P_\infty\{\psi^{-1}[\psi(a) \cdot \alpha(-1/x)]\}$;

$$\begin{aligned}\gamma[P_\infty(r)] &= (g/[\psi(a) \cdot \alpha(r)], g+d) && \text{for } r > 0, \\ &= (g/[a \cdot \alpha(pr)], g+d) && \text{for } r < 0;\end{aligned}$$

and

$$\begin{aligned}x' &= (g/a) \cdot \alpha\{x/[p \cdot (y+k+1)]\} && \text{for } x \geq 0 \text{ and } (y+k+1) < 0, \\ &= [g/\psi(a)] \cdot \alpha[x/(y+k+1)] && \text{for } x \cdot (y+k+1) > 0, \\ &= (g/a) \cdot \alpha[x/(y+k+1)] && \text{for } x \leq 0 \text{ and } (y+k+1) > 0.\end{aligned}$$

PROOF. (A) The general affine isomorphism β is given by [4, p. 54, Theorem 1], with $\psi(e) = e = 0$ as in Theorem 1 above.

(B) The general nonaffine isomorphism γ , according to [5, p. 650, Theorem 3_a], is $\tau\delta\beta$; where τ is a translation $(x, y) \rightarrow (x, y+k)$, δ [called " α " in Theorem 3] is a special collineation of $M_\phi(F)$, and β is the general affine collineation (Part (A) above).

The collineation $\tau\delta = \sigma$ (which reduces to δ in the special case

$k=0$) can be obtained from [5, p. 655, Corollary 3] by putting $a=h=0$ and $p=1$, replacing y by $y+k$, and finally changing q to p .

(C) The explicit γ is obtained by calculating the resultant of σ followed by β .

The proof is now complete.

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