HOMOTOPY GROUPS FOR CELLULAR SET-VALUED FUNCTIONS

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Introduction. Let X be a compact metric space, and 2^X be the collection of all nonempty closed subsets of X . The m th homotopy group $M\Pi_m(2^X, A)$ for set-valued functions has been studied in [3]. By applying the results of [1], we will prove that if $W(n)$ is the set of all cellular subsets of S^n and $p \in S^n$, then $M\Pi_m(W(n), p)$ is isomorphic to the ordinary m th homotopy group of S^n .

Let I^m be the product of m closed unit intervals, and $\text{Bd } I^m$ be the boundary of I^m . Each point $x \in I^m$ is represented by $x = (x_1, \dots, x_m)$, $0 \leq x_i \leq 1$, $i = 1, \dots, m$. Let W be a nonempty subset of 2^X and $A \in W$. We define $\Gamma^m(W, A)$ to be the set of all upper semicontinuous functions F on I^m such that for each $x \in I^m$, $F(x) \in W$; and if $x \in \text{Bd } I^m$, then $F(x) = A$. If $F, G \in \Gamma^m(W, A)$, then we say that F is M -homotopic to G in $\Gamma^m(W, A)$ if there is an upper semicontinuous function H on $I^m \times I$ such that for each $x \in I^m$, $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$; and if $x \in \text{Bd } I^m$, $0 \leq t \leq 1$, then $H(x, t) = A$. This M -homotopy relation is an equivalence relation and hence divides the set $\Gamma^m(W, A)$ into the equivalence classes. We denote the class of M -homotopic to F by $[F]$. For $F, G \in \Gamma^m(W, A)$, we define $H = F + G$ by

$$\begin{aligned} H(x_1, x_2, \dots, x_m) &= F(2x_1, x_2, \dots, x_m) & \text{if } 0 \leq x_1 \leq \frac{1}{2}, \\ &= G(2x_1 - 1, \dots, x_m) & \text{if } \frac{1}{2} \leq x_1 \leq 1, \end{aligned}$$

and $[F] \oplus [G] = [F + G]$. Then $[F + G]$ is independent of representatives of $[F]$ and $[G]$. The proof of the following is similar to that of [3, Theorem 5].

THEOREM 1. *The set of M -homotopy classes in $\Gamma^m(W, A)$ forms a group $M\Pi_m(W, A)$ called the m th homotopy group of W with base A . The zero element in $\Gamma^m(W, A)$ is the constant function $F(x) = A$ for all $x \in I^m$, and the inverse of a function F is defined by $F^{-1}(x_1, \dots, x_m) = F(1 - x_1, \dots, x_m)$.*

Let S_n be an n -sphere. A subset A of S^n is said to be cellular if and only if there is a sequence $E_1 \supset E_2 \supset \dots$ of topological n -cells such that $A = \bigcap_{k=1}^{\infty} E_k$ and $A \subset \text{interior } E_k$ for each $k = 1, 2, \dots$. Let $W(n)$ be the set of all cellular subsets of S^n . Throughout we take S^n to be the unit n -sphere.

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LEMMA 1. Let X be S^m (or the quotient space $S^m \times I/q \times I$, $q \in S^m$), G be a finite open covering of X , and $a \in X$. Then there exists a finite open covering G^* of X of order m (or $m+1$) which is a star refinement of G and the point a belongs to one and only one member of G^* .

PROOF. See [2, Theorem V.1].

By applying Lemma 1 to the proof of [1, Theorem 1], we obtain the following:

LEMMA 2. Let $p \in S^n$ and F be an upper semicontinuous function on X such that $F(x) \in W(n)$ for each $x \in X$ and $F(a) = p$. Then there exists a single-valued continuous function g on X into S^n such that $g(x) \in S^n - F(x)$ for each $x \in X$ and $g(a) = -p$ where $-p$ is the antipodal point of p .

We define $F^m(S^n, p)$ to be the set of all single-valued continuous functions f on I^m into S^n such that $f(x) = p$ for each $x \in \text{Bd } I^m$. Since each point of S^n is cellular and continuous functions are also upper semicontinuous, we consider each $f \in \Gamma^m(S^n, p)$ as an element of $\Gamma^m(W(n), p)$.

LEMMA 3. For each $F \in \Gamma^m(W(n), p)$ there is an element $f \in F^m(S^n, p)$ such that F is M -homotopic to f in $\Gamma^m(W(n), p)$.

PROOF. Let $r: I^m \rightarrow S^m$ be a single-valued continuous function which carries $(I^m - \text{Bd } I^m)$ homeomorphically onto $S^m - \{a\}$ and $r(x) = a$ for each $x \in \text{Bd } I^m$. Let $G(x) = F \circ r^{-1}(x)$. Then G is upper semicontinuous on S^m and $G(x) \in W(n)$ for each $x \in S^m$. Let $g^*: S^m \rightarrow S^n$ be a single-valued continuous function which is provided by Lemma 2 for G , and let $g = g^* \circ r$ and $f(x) = -g(x)$. Then $f \in F^m(S^n, p)$. Now for each $b \in S^n$, let $J_b: [S^n - b] \times I \rightarrow S^n$ be defined by

$$J_b(x, t) = (-tb + (1-t)x) / \|-tb + (1-t)x\|;$$

and we let $H(x, t) = \{J_{\sigma(x)}(y, t) \mid y \in F(x)\}$. Then H is an M -homotopy in $\Gamma^m(W(n), p)$ relating F to f .

LEMMA 4. If $f_0, f_1 \in F^m(S^n, p)$ and H is an M -homotopy in $\Gamma^m(W(n), p)$ relating f_0 to f_1 , then there exists an ordinary homotopy h in $F^m(S^n, p)$ relating f_0 to f_1 .

PROOF. Let $q \in S^n$, $X = S^m \times I/q \times I$, and $a = q \times I$. We define a mapping $r: I^m \times I \rightarrow X$ be a single-valued continuous function which carries $(I^m \times I - \text{Bd } I^m \times I)$ homeomorphically onto $X - \{a\}$ and $r(x, t) = a$ for each $x \in \text{Bd } I^m$ and $0 \leq t \leq 1$. Let $T(z) = H \circ r^{-1}(z)$ for each $z \in X$. Then T is upper semicontinuous on X and $T(z) \in W(n)$ for each $z \in X$, and $T(a) = p$. Let $g^*: X \rightarrow S^n$ be a single-valued continuous function which is provided by Lemma 2 for T , and define

$g(x, t) = g^* \circ r(x, s)$, $K(x, s, t) = \{J_{\sigma(x, s)}(y, t) \mid y \in H(x, s)\}$, and $\tilde{K}(x, s, t) = K(r(x, s), t)$. We define an ordinary homotopy h in $F^m(S^n, p)$ by

$$\begin{aligned} h(x, t) &= \tilde{K}(x, 0, 3t) && \text{if } 0 \leq t \leq 1/3, \\ &= \tilde{K}(x, 3t - 1, 1) && \text{if } 1/3 \leq t \leq 2/3, \\ &= \tilde{K}(x, 1, 3 - 3t) && \text{if } 2/3 \leq t \leq 1. \end{aligned}$$

Then $h: I^m \times I \rightarrow S^n$ is a single-valued upper semicontinuous function and hence h is continuous.

THEOREM 2. $M\Pi_m(W(n), p)$ is isomorphic to the ordinary m th homotopy group $\Pi_m(S^n, p)$ of S^n .

PROOF. For each $F \in \Gamma^m(W(n), p)$, let $f \in F^m(S^n, p)$ be an element which is given by Lemma 3, and define $\lambda: M\Pi_m(W(n), p) \rightarrow \Pi_m(S^n, p)$ by $\lambda[F] = (f)$ where (f) is the ordinary homotopy class in $F^m(S^n, p)$ containing f . If G is a member of $[F]$ and $g \in F^m(S^n, p)$ a corresponding element for G , then f and g are M -homotopic. Hence by Lemma 4, f and g are homotopic in $F^m(S^n, p)$. Therefore, λ is well defined. Let (f) be an element of $\Pi_m(S^n, p)$ which is represented by f . Since f is also an element of $\Gamma^m(W(n), p)$, there is an element $[F]$ in $M\Pi_m(W(n), p)$ such that $f \in [F]$ and $\lambda[F] = (f)$. That λ is an isomorphism follows from Lemma 4 and the definition of $\oplus$.

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