SOME RESULTS FOR BOREL TRANSFORMS

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1. Introduction. Let $\lambda = \{\lambda_p\}$ be a sequence of complex numbers and let the matrix $A(\lambda) = (a_{nk})$ be defined by

$$a_{0k} = 0, \quad k = 0, 1, 2, \dots$$

$$(1) \quad \prod_{p=1}^n \exp(\lambda_p(z-1)) = \sum_{k=0}^{\infty} a_{nk} z^k.$$

In the special case $\lambda = \delta = (1, 1, 1, \dots)$, the elements of $A(\lambda) = A(\delta)$ reduce to $a_{nk} = e^{-n} n^k / k!$. The matrix $A(\delta)$ is the discrete Borel sequence-to-sequence transform which has been studied by Gaier [3] and Meyer-König and Zeller [7].

In §2 the regularity of the matrix $A(\lambda)$ is investigated and a theorem concerning the summability of Taylor series by $A(\lambda)$ is considered. In §3 the $A(\lambda)$ -summability of series of Legendre polynomials is studied. These results are believed to be new even in the case $\lambda = \delta$. In §4 a theorem concerning the approximation properties of a class of positive linear operators, defined on $C[0, b]$ which are associated with $A(\lambda)$ is proved.

These operators are generalizations of operators which have been studied by Szász [8], Cheney and Sharma [1], and G. M. Mirakyan [5, p. 53].

2. Regularity of $A(\lambda)$.

THEOREM 1. *If $\sum_{p=1}^n \lambda_p \geq 0$ for each $n = 1, 2, \dots$, and $\sum_{p=1}^{\infty} \lambda_p = \infty$ then $A(\lambda)$ is regular.*

PROOF. It is clear that $\sum_{k=0}^{\infty} a_{nk} = 1$. Since

$$a_{nk} = \frac{1}{k!} \exp\left(-\sum_{p=1}^n \lambda_p\right) \cdot \left(\sum_{p=1}^n \lambda_p\right)^k, \quad k, n = 1, 2, \dots,$$

it is clear that $a_{nk} \geq 0$ for $k, n = 0, 1, 2, \dots$, and $\lim_{n \rightarrow \infty} a_{nk} = 0$ for $k = 0, 1, 2, \dots$. The Silverman-Toeplitz conditions are, therefore, satisfied. Hence $A(\lambda)$ is regular.

THEOREM 2. *If λ_p is real for $p = 1, 2, \dots$ and $\sum_{p=1}^{\infty} \lambda_p = \infty$ then the sequence of partial sums of the geometric series is $A(\lambda)$ -summable to $1/(1-z)$ in the half-plane $H = \{z: \operatorname{Re}(z) < 1\}$. The summability is uniform on each compact subset of H .*

Received by the editors April 12, 1967.

PROOF. Let $\{\sigma_n\}$ denote the A -transform of the sequence of partial sums of the geometric series. Let

$$R = \left\{ z: \lim_{n \rightarrow \infty} \sigma_n = 1/(1-z) \right\}.$$

It is easy to see that

$$R = \left\{ z: \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} z^k = 0 \right\}.$$

But

$$\sum_{k=0}^{\infty} a_{nk} z^k = \prod_{p=1}^n \exp(\lambda_p(z-1))$$

for $n = 1, 2, \dots$. Hence

$$\left| \sum_{k=0}^n a_{nk} z^k \right| = \exp \left(\operatorname{Re}(z-1) \sum_{p=1}^n \lambda_p \right).$$

So $z \in R$ if and only if $\operatorname{Re}(z) < 1$. Hence $R = H$.

THEOREM 3. *Let λ_p be real for $p = 1, 2, \dots$ and let $\sum_{p=1}^{\infty} \lambda_p = \infty$. If $\sum a_n z^n$ is a Taylor series with nonzero radius of convergence then the sequence of partial sums is $A(\lambda)$ -summable to the value of the function defined by the series in the Borel polygon of the function.*

PROOF. The proof is a straightforward application of Okada's theorem [6, p. 119].

3. Summability of series of Legendre polynomials. Let P_n and Q_n denote, respectively, the n th Legendre polynomial of first and second kind. P_n and Q_n are determined by [9, Chapter XV],

$$(2) \quad P_n = \frac{1}{\pi} \int_0^{\pi} [\zeta(\phi)]^n d\phi,$$

and

$$(3) \quad Q_n = \int_0^{\infty} [\tau(u)]^{-n-1} du,$$

where $\zeta(\phi) = z + (z^2 - 1)^{1/2} \cos \phi$ and $\tau(u) = t + (t^2 - 1)^{1/2} \cosh u$. (The branch of $(z^2 - 1)^{1/2}$ is chosen so that $z + (z^2 - 1)^{1/2}$ lies outside the unit circle.)

Let $\{s_n(z, t)\}$ be defined by

$$s_n(z, t) = \sum_{k=0}^n (2k+1) P_k(z) Q_k(t), \quad n = 0, 1, \dots$$

It is known [9, Chapter XV] that $\{s_n(z, t)\}$ converges to $1/(t-z)$ if t is in the interior of the ellipse with foci ± 1 which passes through t . The following theorem shows that, under certain conditions, $\{s_n(z, t)\}$ is $A(\lambda)$ -summable to $1/(t-z)$ in a larger region.

THEOREM 4. *Let λ_p be real for $p=1, 2, \dots$ and $\sum_{p=1}^{\infty} \lambda_p = \infty$. Let $0 < \alpha < 1$, $0 < \beta$ and let*

$$S(t, \alpha) = \{z: \operatorname{Re}(\zeta(\phi)/\tau(u)) < \alpha, 0 \leq \phi \leq \pi, 0 \leq u\},$$

$$T(t, \beta) = \{z: |\zeta(\phi)/\tau(u)| < \beta, 0 \leq \phi \leq \pi, 0 \leq u\},$$

and

$$R(t, \alpha, \beta) = S(t, \alpha) \cap T(t, \beta).$$

Then $\{s_n(z, t)\}$ is $A(\lambda)$ -summable to $1/(t-z)$ if $z \in R(t, \alpha, \beta)$.

PROOF. It follows from the Christoffel formula [9, Chapter XV] that

$$(4) \quad 1/(t-z) = s_n(z, t) + (n+1)d_n(z, t)/(t-z)$$

where

$$(5) \quad d_n(z, t) = P_{n+1}(z)Q_n(t) - P_n(z)Q_{n+1}(t).$$

It follows from (2), (3), and (5) that

$$(6) \quad d_n(z, t) = \frac{1}{\pi} \int_0^\pi \int_0^\infty \left(\frac{\zeta}{\tau}\right)^n \left[\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right] d\phi du.$$

So if $\{\sigma_n(z, t)\}$ denotes the $A(\lambda)$ -transform of $\{s_n(z, t)\}$, it follows from (4), (5) and (6) that

$$1/(t-z) = \sigma_n(z, t) + h_n(z, t)/(t-z),$$

where

$$h_n(z, t) = \sum_{k=0}^{\infty} a_{nk} \frac{(k+1)}{\pi} \int_0^\pi \int_0^\infty \left(\frac{\zeta}{\tau}\right)^k \left[\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right] d\phi du.$$

Hence, $\lim_{n \rightarrow \infty} \sigma_n(z, t) = 1/(t-z)$ if $\lim_{n \rightarrow \infty} h_n(z, t) = 0$.

It is not difficult to see that $\int_0^\pi \int_0^\infty \{|\zeta/\tau| + |1/\tau^2|\} d\phi du$ is uniformly bounded for $z \in R(t, \alpha, \beta)$ so that $\int_0^\pi \int_0^\infty [\zeta/\tau - 1/\tau^2] d\phi du$ is uniformly bounded on $R(t, \alpha, \beta)$.

A formal interchange of integration and summation gives

$$h_n(z, t) = \frac{1}{\pi} \int_0^\pi \int_0^\infty \left[\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right] \sum_{k=0}^{\infty} (k+1) a_{nk} \left(\frac{\zeta}{\tau}\right)^k d\phi du.$$

A simple calculation then yields

$$h_n(z, t) = \frac{1}{\pi} \int_0^\pi \int_0^\infty \left[\frac{\zeta}{\tau} - \frac{1}{\tau^2} \right] \cdot \prod_{p=1}^n \exp \left(\lambda_p \left(\frac{\zeta}{\tau} - 1 \right) \right) \left[1 + \frac{\zeta}{\tau} \sum_{p=1}^n \lambda_p \right] d\phi du.$$

Let

$$I_n = \prod_{p=1}^n \exp \left(\lambda_p \left(\frac{\zeta}{\tau} - 1 \right) \right) \left[1 + \frac{\zeta}{\tau} \sum_{p=1}^n \lambda_p \right].$$

It follows that

$$|I_n| = \exp \left(\operatorname{Re} \left(\frac{\zeta}{\tau} - 1 \right) \sum_{p=1}^n \lambda_p \right) \left| 1 + \frac{\zeta}{\tau} \sum_{p=1}^n \lambda_p \right|.$$

Hence $z \in R(t, \alpha, \beta)$ implies $\lim_{n \rightarrow \infty} I_n = 0$ and therefore $\lim_{n \rightarrow \infty} h_n(z, t) = 0$. Moreover, if $z \in T(t, \beta)$ the interchange of integration and summation used in the above is valid since the series involved converges uniformly. Therefore, $z \in R(t, \alpha)$ implies $\lim_{n \rightarrow \infty} \sigma_n(z, t) = 1/(t-z)$. This completes the proof.

The sets $S(t, \alpha)$ and $T(t, \beta)$ are the same as regions which were studied by Cowling and King [2] in connection with the Lototsky and Taylor summability of series of Legendre polynomials. An analysis identical to that used in [2] gives the following result

THEOREM 5. *Let λ_p be real for $p=1, 2, \dots$ and let $\sum_{p=1}^\infty \lambda_p = \infty$. Let $0 < \alpha < 1$ and $0 < \beta$. For each $u \geq 0$, let $H(\tau(u), \alpha)$ be the open half-plane containing the origin whose boundary is the line through the point $\alpha\tau(u)$ which is perpendicular to the line from the origin through $\tau(u)$. Let $B(t)$ be the image of $\cap \{H(\tau(u), \alpha): u \geq 0\}$ under the mapping $t=1/2(w+1/w)$. Let $D(t)$ denote the bounded region whose boundary is the ellipse with foci ± 1 which passes through the point $1/2[(\beta+1/\beta)t + (\beta-1/\beta)(t^2-1)^{1/2}]$. Then $\{s_n(z, t)\}$ is $A(\lambda)$ -summable to $1/(t-z)$ whenever z lies in any compact subset of $B(t) \cap D(t)$.*

Now let f be analytic on the interval $[-1, 1]$ and let $\{t_n(z)\}$ be the sequence of partial sums of the series of Legendre polynomials which represents $f(z)$ in the ellipse with foci ± 1 which passes through no singular point of f . The region in which $\{t_n(z)\}$ is $A(\lambda)$ -summable to $f(z)$ may be determined in terms of $B(t) \cap D(t)$ and the distribution of the singular points of f by an analysis identical to that used in [2]. This analysis gives the result

THEOREM 6. Let λ_p be real for $p=1, 2, \dots$ and let $\sum_{p=1}^{\infty} \lambda_p = \infty$. Let $0 < \alpha < 1$ and $0 < \beta$. Let $B(t)$ and $D(t)$ be as in Theorem 5 and let $V(t) = B(t) \cap D(t)$. The sequence $\{t_n(z)\}$ is $A(\lambda)$ -summable to $f(z)$ whenever z lies in any compact subset of $\cap \{V(t): t \in U\}$ where U is the set of all singular points of f .

It is shown in [2] that the region $\cap \{V(t): t \in U\}$ contains the ellipse of convergence of $\{t_n(z)\}$. Theorem 6 may also be obtained from Theorem 5 by making use of a general result of Jakimovski's [4].

4. A positive linear operator associated with $A(\lambda)$. Let $b > 0$ and let λ_p be a real valued function defined on the interval $[0, b]$. Let B denote the real valued functions which are bounded on $[0, \infty)$. Let $a_{nk} = a_{nk}(x)$ be defined by:

$$(7) \quad \begin{aligned} a_{0k} &= 0, & k &= 0, 1, 2, \dots, \\ \prod_{p=1}^n \exp(\lambda_p(x)(z-1)) &= \sum_{k=0}^{\infty} a_{nk}(x)z^k. \end{aligned}$$

For each $f \in C[0, b] \cap B$ let

$$(8) \quad B_n(f)(x) = \sum_{k=0}^{\infty} a_{nk}(x)f\left(\frac{k}{n}\right).$$

THEOREM 7. Let $\{t_n(x)\}$ denote the $(C, 1)$ -transform of the sequence $\{\lambda_p(x)\}$. If $\lim_{n \rightarrow \infty} t_n(x) = x$ uniformly on $[0, b]$, and if $\sum_{p=1}^n \lambda_p(x) \geq 0$ for each $n=1, 2, \dots$, and $x \in [0, b]$, then $\lim_{n \rightarrow \infty} B_n(f)(x) = f(x)$ uniformly on $[0, b]$ for each $f \in C[0, b] \cap B$.

PROOF. The hypothesis $\sum_{p=1}^n \lambda_p(x) \geq 0$ implies that $a_{nk}(x) \geq 0$ for each $x \in [0, b]$. Hence $f(x) \geq 0$ for $x \in [0, b]$ implies that $B_n(f)(x) \geq 0$ for each $x \in [0, b]$. Therefore B_n is a positive operator on $C[0, b]$ for each $n=1, 2, \dots$. It is clear that B_n is linear for each $n=1, 2, \dots$. According to a theorem of Korovkin [5, pp. 7, 8] it is sufficient to prove that $\lim_{n \rightarrow \infty} B_n(e^i)(x) = x^i$ uniformly on $[0, b]$ for $i=0, 1, 2$, where e^i is defined by $e^i(x) = x^i$ for $i=0, 1, 2$.

It is clear that $B_n(e^0)(x) = 1$. Let $P_n(z) = \prod_{p=1}^n \exp(\lambda_p(x)(z-1))$. It follows from (7) that

$$P'_n(z) = \sum_{k=0}^{\infty} k a_{nk}(x) z^{k-1}.$$

But also

$$(9) \quad P'_n(z) = P_n(z) \sum_{p=1}^n \lambda_p(x).$$

The last two equations, and the fact that $P_n(1) = 1$, imply

$$\frac{1}{n} \sum_{k=0}^{\infty} k a_{nk}(x) = \frac{1}{n} \sum_{p=1}^n \lambda_p(x).$$

Hence $B_n(e^1)(x) = t_n(x)$. Therefore

$$\lim_{n \rightarrow \infty} B_n(e^1)(x) = x$$

uniformly on $[0, b]$. It follows from (7) that

$$P_n''(z) = \sum_{k=0}^{\infty} k(k-1) a_{nk}(x) z^{k-2},$$

and from (9) that

$$P_n''(z) = \left[\sum_{p=1}^n \lambda_p(x) \right]^2 \prod_{p=1}^n \exp(\lambda_p(x)(z-1)).$$

Therefore

$$\left[\sum_{p=1}^n \lambda_p(x) \right]^2 = \sum_{k=0}^{\infty} k^2 a_{nk}(x) - \sum_{k=0}^{\infty} k a_{nk}(x),$$

and hence

$$B_n(e^2)(x) = \frac{1}{n^2} \left[\sum_{p=1}^n \lambda_p(x) \right]^2 + \frac{1}{n^2} \sum_{k=0}^{\infty} k a_{nk}(x).$$

Therefore,

$$B_n(e^2)(x) = t_n^2(x) + t_n(x)/n.$$

Consequently,

$$\lim_{n \rightarrow \infty} B_n(e^2)(x) = x^2$$

uniformly on $[0, b]$. This proves the theorem.

COROLLARY. *Let $\lambda_p(x) \geq 0$ for each $x \in [0, b]$ and $p = 1, 2, \dots$. If $\{\lambda_p(x)\}$ converges to x uniformly on $[0, b]$ then $\lim_{n \rightarrow \infty} B_n(f, x) = f(x)$ uniformly on $[0, b]$ for each $f \in C[0, b] \cap B$.*

PROOF. The $(C, 1)$ method is regular and preserves uniform convergence.

In the special case $\lambda_p(x) = x$, $p = 1, 2, \dots$, the operators B_n reduce to

$$B_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} f\left(\frac{k}{n}\right).$$

For this choice of $\lambda_p(x)$, Theorem 5 gives a result due to Szasz [8] and, evidently, to Mirakyan [5, p. 53].

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