ON MONOTONE UNIONS OF MANIFOLDS

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If U is a connected separable metric space that is locally euclidean of dimension n , U is an n -manifold. We do distinguish n -manifolds and n -manifolds with boundary. By $\mathfrak{M}(U)$ we denote the collection of all n -manifolds M^n such that $M^n = \bigcup_{i=1}^{\infty} U_i$, where $U_i \subset U_{i+1}$ and each U_i is a topological copy of U . The size of the collection is quite variable and we list some examples from which obvious questions arise.

1. If S^2 is the 2-sphere and E^1 the real line, let $U = S^2 \times E^1$. In this case $\mathfrak{M}(U)$ has two members, U and E^3 , where E^3 is euclidean 3-space.

2. Let M^n be a combinatorial compact n -manifold with $n \neq 4$. If U is the complement of a flat n -cell in M^n , then $\mathfrak{M}(U)$ has U as its only member [5]. The same result when U is a cell follows from [2].

3. If U is S^2 with an infinite convergent sequence of points removed, then $\mathfrak{M}(U)$ consists of exactly the proper open connected sets (proper domains) in S^2 .

A manifold such as U in example 2 above for which $\mathfrak{M}(U)$ consists of one element is said to have the monotone union property. If M^n is an n -manifold, then a standard decomposition of M^n is a representation of M^n as a disjoint union, $M^n = P \cup R$, where $P = E^n$ and $\dim R \leq n-1$ [3].

THEOREM 1. *Let M^n be a compact n -manifold and $M^n = P \cup R$ a standard decomposition of M^n . If $U \supset R$ is any proper topological n -manifold in M^n , then U has the monotone union property only if $U = M^n - x$, where x is a point in M^n .*

PROOF. Suppose $U \supset R$ is given and U has the monotone union property. Let x be a point in $M^n - U$. Then if we select a small open n -cell about x , C^n , with a flat boundary it follows that there is a homeomorphism of $M^n - x$ into $M^n - x$ that is fixed outside $\text{Bd } C^n$ and the image fails to meet a neighborhood of x in M^n . Thus we may assume that U is not dense in M^n .

By the construction in [3] it follows that there is a bicollared $(n-1)$ -sphere S^{n-1} in U , S^{n-1} is the boundary of a flat n -cell K in P that contains the point x in its interior. If one now places a flat n -cell K^1 about x in $\text{Int } K^1$, $K^1 \subset M^n - U$, it can be seen that $M^n - x$ is a monotone union of copies of U . Thus if U has this property $U = M^n - x$.

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THEOREM 2. *Let M^n be a topological n -manifold and $C \neq \emptyset$ a compact proper subset of M^n . If $M^n - C$ is a manifold while C lies in an open n -cell in M^n , then $M^n - C$ has the monotone union property if and only if M^n less a point has this property and C is cellular in M^n .*

PROOF. The sufficiency is trivial, let $M^n - C$ have the monotone union property. By hypothesis we may assume that C lies in the interior of a flat n -cell C^n in M^n .

Now if C is not cellular in M^n , let A be a tame arc in $\text{Int } C^n$ with endpoints a and b , while $A \cap C = a$. At b introduce a flat n -cell K so that $K \cap A = b$, $K \cap C = \emptyset$, $K \subset \text{Int } C^n$, and $K \cup A$ is a tame set. If $C^1 = C \cup A \cup K$, we have $M^n - C = M^n - C^1$; there is a homeomorphism h_1 of M^n onto M^n such that $h_1(C^n) = K$ while the diameter of $h_1(C^1)$ $< \frac{1}{2}$. By repeating the construction it is clear that by induction we obtain M^n less a point as a monotone union of copies of $M^n - C$. So $M^n - C = M^n - x$, where x is a point and C is cellular.

In [5] the sets considered are piecewise linear compact n -manifolds with $(n-1)$ -sphere boundaries. For $n \neq 4$ it is shown that their interiors have the monotone union property. The cells among these spaces have the property that each carries a pair of disjoint copies of itself. We show that this is a peculiarity of cells among the spaces for $n \geq 5$.

LEMMA 1. *Let M^n be a compact n -manifold with an $(n-1)$ -sphere boundary, $n \geq 3$. If in $\text{Int } M^n$ there is a topological copy T^1 of M^n , then in $\text{Int } T^1$ there is a topological copy T of M^n and $X = M^n/T$ is a simply connected n -manifold with an $(n-1)$ -sphere boundary.*

PROOF. In T^1 , $\text{Bd } T^1$ is collared [1]. Thus we may select a sphere in this collar that bounds a copy T of M^n in $\text{Int } T^1$, $\text{Bd } T$ is bicollared; then $X = M^n/T$ is an n -manifold with an $(n-1)$ -sphere boundary. If $\pi_1(X) \neq 1$, we note that by the Van Kampen theorem

$$\pi_1(M^n) \cong \pi_1(\overline{M^n - T}) * \pi_1(T),$$

the free product. But

$$\pi_1(X) \cong \pi_1(\overline{M^n - T}).$$

Clearly $\pi_1(M^n)$ is then not finitely generated.

COROLLARY. *If $\pi_1(M^n) \neq 1$, M^n cannot contain two disjoint copies of itself.*

By these observations we note that if M^n is as in Lemma 1 and contains two disjoint copies of itself, then $\pi_1(M^n) = 1$.

We conclude that M^n is homologically trivial by an argument on homology analogous to the above argument on $\pi_1(M^n)$.

THEOREM 3. *Let $M^n \neq 4$ be a compact topological n -manifold with an $(n-1)$ -sphere boundary. If M^n contains two disjoint topological copies of itself, M^n is a closed n -cell.*

PROOF. Since M^n is contractible, if one sews an n -cell into M^n by a homeomorphism on the boundary of the cell to that of M^n , the result is a n -sphere by [6] for $n \neq 3$ or 4. Thus M^n is an n -cell. In the case $n = 3$, the Theorem follows from [4].

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