CONCERNING WILD CANTOR SETS IN E^3

R. B. SHER[1]

1. **Introduction.** A well-known example of a wild Cantor set in E^3 is that described by Antoine in [2]. This example has become known as Antoine's necklace. In this paper we introduce the concept of defining sequences for Cantor sets in E^3. Using defining sequences, we define a large class of Cantor sets in E^3, to be referred to as Antoine's necklaces, which are closely related to the original example of Antoine. A necessary and sufficient condition is given for two of these Cantor sets to be equivalently imbedded in E^3. Finally, we conclude with a result that shows the existence of uncountably many inequivalent upper semicontinuous decompositions of E^3.

If A is a subset of E^3, we shall use Cl A, Bd A, and Int A to denote the closure, boundary and interior, respectively, of A. Interior may refer either to the combinatorial interior of a manifold with boundary or to the bounded complementary domain of a compact 2-manifold in E^3; the correct interpretation will be clear in each case from context.

For terms not defined in this paper (particularly those dealing with linked chains and winding) one may refer to [11].

2. **Defining sequences.** A sequence M_1, M_2, M_3, $\cdots$ of compact 3-manifolds with boundary is a *defining sequence* for the Cantor set M in E^3 if and only if (1) for each positive integer i, M_i is the union of a finite number of mutually exclusive polyhedral cubes with handles, (2) for each positive integer i, $M_{i+1} \subset$ Int M_i, and (3) $M = \bigcap_{i=1}^{\infty} M_i$. One may compare this definition to that of a defining sequence for an upper semicontinuous decomposition of E^3, [4]. It is known ([3] or [10]) that every Cantor set in E^3 has a defining sequence. Two such sequences M_1, M_2, M_3, $\cdots$ and N_1, N_2, N_3, $\cdots$ are said to be *equivalent* if for each positive integer i there is a homeomorphism h_i of E^3 onto itself such that (1) for each positive integer i, $h_{i+1} | (E^3 - M_i) = h_i | (E^3 - M_i)$, and (2) for each positive integer i, $h_i(M_i) = N_i$. It is not difficult to show that this notion yields an equivalence relation on the class of defining sequences of cantor sets in E^3. We shall use the notation $\{M_i\}$ to denote the defining sequence M_1, M_2, M_3, $\cdots$ and $\{M_i\} \sim \{N_i\}$ to denote that $\{M_i\}$ and $\{N_i\}$ are equivalent defining sequences.

The following theorem may be compared with Theorem 3 of [6].

THEOREM 1. *The Cantor sets M and N are equivalently imbedded in E^3 if and only if there exist defining sequences $\{M_i\}$ and $\{N_i\}$ for M and N respectively with $\{M_i\} \sim \{N_i\}$.*

PROOF. Suppose first of all that M and N are equivalently imbedded in E^3 and let h be a homeomorphism of E^3 onto itself carrying M onto N.

Let $\{M_i\}$ be a defining sequence for M. If for each positive integer i we let $N_i = h(M_i)$ and $h_i = h$, it is easily seen that $\{N_i\}$ is a defining sequence for N equivalent to $\{M_i\}$.

Now suppose M and N possess equivalent defining sequences $\{M_i\}$ and $\{N_i\}$ respectively, where the equivalence is shown by the sequence of homeomorphisms $\{h_i\}$. We define the function h from E^3 onto itself as follows: If $x \in E^3 - M$, let i be an integer such that $x \notin M_i$ and let $h(x) = h_i(x)$. If $x \in M$, let $h(x) = \lim_{i \to \infty} h_i(x)$. The function h so defined is a homeomorphism of E^3 onto itself and $h(M) = N$.

Unfortunately, each Cantor set in E^3 has many nonequivalent defining sequences; so one is able to use the above theorem to distinguish between two imbeddings of the Cantor set in E^3 only if one compares all possible defining sequences for each. In the following section we shall define a collection of Cantor sets, each with a canonical defining sequence which one can use to distinguish imbeddings.

3. Antoine's necklace.

The examples described in this section are suggested by the wild Cantor set, often referred to as Antoine's necklace, described by Antoine in [2].

Simple closed curves J_1 and J_2 in E^3 are said to be of *simple linking type* if and only if there is a homeomorphism h of E^3 onto itself such that $h(J_1) = \{(x, y, z) \mid x^2 + y^2 = 1, \ z = 0\}$ and $h(J_2) = \{(x, y, z) \mid (y-1)^2 + z^2 = 1, \ x = 0\}$. Note that J_1 and J_2 are of simple linking type if and only if J_2 and J_1 are of simple linking type. Disjoint solid tori T_1 and T_2 in E^3 are of *simple linking type* if and only if the centers of T_1 and T_2 are of simple linking type. A linked chain of $n \geq 3$ solid tori is of *simple chain type* if and only if each linked pair of solid tori in the chain is of simple linking type.

Now, let T be a solid torus in E^3. Let $Z: T_1, \cdots, T_n$ $(n \geq 3)$ be a linked chain of simple chain type in Int T with winding number, $W(Z, T) > 0$, and let $M_1 = \bigcup T_i$. Let $Z_i: T_{i1}, \cdots, T_{in_i}$ $(n_i \geq 3)$ be a linked chain of simple chain type in Int T_i with $W(Z_i, T_i) > 0$ and let $M_2 = \bigcup T_{ij}$. Let $Z_{ij}: T_{ij1}, \cdots, T_{ijn_{ij}}$ $(n_{ij} \geq 3)$ be a linked chain of simple chain type in Int T_{ij} with $W(Z_{ij}, T_{ij}) > 0$, let $M_3 = \bigcup T_{ijk}$, and so on. This is to be done in such a way that if d_i denotes the maximum of the set of diameters of components of M_i, then $d_i \to 0$ as $i \to \infty$. Then

$M = \bigcap_{i=1}^{\infty} M_i$ is a Cantor set with defining sequence $\{M_i\}$. Under these conditions, we shall say that M is an *Antoine's necklace* with *canonical defining sequence* $\{M_i\}$. The usual Antoine's necklace that has been described in the literature is one in which $n_{ij\ldots k} = 4$ and $W(Z_{ij\ldots k}, T_{ij\ldots k}) = 1$.

The Antoine's necklaces have some interesting properties. They are all wildly imbedded in E^3. If M is an Antoine's necklace with defining sequence $\{M_i\}$, i is a positive integer, and J is a simple closed curve on Bd M_i which bounds no disk on Bd M_i, then J cannot be shrunk to a point in $E^3 - M$. Also, M is not separated in E^3 by any 2-sphere.

The following theorem shows that the canonical defining sequences determine the imbedding classes of the Antoine's necklaces. I am told that a similar theorem has also been proven by Professors Steve Armentrout and R. H. Bing.

THEOREM 2. *Suppose M and N are Antoine's necklaces in E^3 with canonical defining sequences $\{M_i\}$ and $\{N_i\}$ respectively. Then M and N are equivalently imbedded in E^3 if and only if $\{M_i\} \sim \{N_i\}$.*

PROOF. The "if" portion of the conclusion follows from Theorem 1. Suppose then that M and N are equivalently imbedded in E^3. It clearly suffices to suppose that $M = N$; that is, that $\{M_i\}$ and $\{N_i\}$ are defining sequences for the same Cantor set. Recalling that M_1 and N_1 are polyhedral, we may further suppose that Bd M_1 is in general position with respect to Bd N_1. This guarantees that $(\mathrm{Bd}\, M_1) \cap (\mathrm{Bd}\, N_1)$ is the union of a finite collection of mutually exclusive simple closed curves $J_1, \cdots, J_q$.

We now describe a sequence of piecewise linear homeomorphisms of E^3 onto itself. The composition of these homeomorphisms, in the order they are described, will be a homeomorphism h_1 of E^3 onto itself which is the identity on M and which carries M_1 onto N_1. We first carry M_1 onto a set whose boundary fails to intersect Bd N_1. This is accomplished, in the following paragraph, by removing the simple closed curves $J_1, \cdots, J_q$ one at a time.

Suppose some $J \in \{J_1, \cdots, J_q\}$ bounds a disk on Bd N_1. We may choose such a J with the property that if D is the disk on Bd N_1 bounded by J, then $(\mathrm{Int}\, D) \cap (\mathrm{Bd}\, M_1) = \varnothing$. Since J can be shrunk to a point in $E^3 - M$, J must bound a disk, say D', on Bd M_1. The interior of the 2-sphere $D \cup D'$ misses M, so there is a homeomorphism of E^3 onto itself, fixed on M, that pushes D' onto D. Pushing to one side or the other of the component of Bd N_1 containing J, we adjust M_1 so that the adjusted Bd M_1 intersects Bd N_1 in at least one less

component than the original. Following this procedure, we adjust M_1 so that the adjusted Bd M_1 contains no simple closed curve that bounds a disk on (Bd M_1)$\cup$(Bd N_1). For ease of notation we shall continue to denote the adjusted M_1 by M_1. Now let J be one of the remaining elements of $\{J_1, \cdots, J_q\}$. There is a $J' \in \{J_1, \cdots, J_q\}$ such that $J \cup J'$ is the boundary of an annulus A on Bd N_1 with (Int A)$\subset$Int M_1. The simple closed curves J and J' lie on the same component, T, of M_1 and Bd $T = A' \cup A''$ where A' and A'' are annuli bounded by $J \cup J'$. The interior of one of the tori $A \cup A'$ and $A \cup A''$ (say, e.g., $A \cup A'$) misses M. Without disturbing M, we push A onto A' and then push slightly further to remove $J \cup J'$ from (Bd M_1)$\cap$(Bd N_1). Continuing this procedure, we adjust M_1 so that (Bd M_1)$\cap$(Bd N_1) $= \varnothing$.

Suppose some component, say T, of M_1 lies in Int N_1 and is null-homotopic in Int N_1. Then, any other component of M_1 linking T must lie in the same component of N_1 as T. But this places all of M in a single component of N_1, a contradiction. Similarly, no component of N_1 lying in Int M_1 is null-homotopic in Int M_1.

Now suppose some component T of N_1 contains more than one component of M_1. The solid torus T is linked by some component of N_1, hence by some component of M_1. By the above paragraph, this component of M_1 links each component of M_1 in T. Hence, there are at least three components of M_1 each of which links a single component of M_1. This is a contradiction, since each solid torus in a linked chain links only two elements of the chain. Similarly no component of M_1 contains more than a single component of N_1.

We have now established a 1-1 correspondence between components of M_1 and N_1. We now show that corresponding components are concentric. Suppose then that T is a component of M_1 containing in its interior T', a component of N_1. We need only show that $W(C, T) = 1$, where C is the center of T'. Let T'' be a component of N_1 so that T' and T'' are of simple linking type. There is a polyhedral disk D with Bd D a longitudinal simple closed curve of Bd T'' and $D \cap C$ a one point set. We may suppose D to be in general position with respect to Bd T. By Theorem 1 of [8], each component of $D \cap$Bd T either bounds a disk on Bd T, or circles Bd T once longitudinally and not at all meridionally, or circles Bd T once meridionally and not at all longitudinally. We use a standard cutting off technique to remove simple closed curves on D which bound disks on Bd T. If some component of $D \cap$Bd T is longitudinal on Bd T, this component can be shrunk to a point in $E^3 - C$, a contradiction since Bd D cannot be shrunk to a point in $E^3 - C$. Hence, the components

of $D \cap \mathrm{Bd}\ T$ are meridional on Bd T and the result is a meridional disk of T which intersects C in a single point. This shows that $W(C, T) = 1$. We now complete the construction of h_1 by shoving M_1 onto N_1 using the fact that components of M_1 and N_1 lie in concentric pairs.

In a manner similar to the above construction we may construct a homeomorphism h_2 of E^3 onto itself which agrees with h_1 outside of M_1, which leaves M fixed, and which moves M_2 onto N_2, $\cdots$, etc. This shows that $\{M_i\} \sim \{N_i\}$.

COROLLARY 1. *There are uncountably many inequivalently imbedded Antoine's necklaces in E^3.*

The final results of this section are concerned with the notion of *equivalent* upper semicontinuous decompositions of E^3. This concept was first introduced in [6], and the reader will find there the appropriate definitions as well as some related results.

Suppose G is an upper semicontinuous decomposition of E^3 with a defining sequence $\{M_i\}$, where $\{M_i\}$ satisfies all of the conditions imposed on a canonical defining sequence for an Antoine's necklace except the condition $d_i \to 0$ as $i \to \infty$. Then G is an *Antoine decomposition* of E^3 with *canonical defining sequence* $\{M_i\}$. An example of an Antoine decomposition of E^3 appears in [5]. The following theorem follows from the proof of Theorem 2 and Theorem 3 of [6].

THEOREM 3. *Suppose G_1 and G_2 are Antoine decompositions of E^3 with canonical defining sequences $\{M_i\}$ and $\{N_i\}$. Then G_1 and G_2 are equivalent if and only if $\{M_i\}$ and $\{N_i\}$ are equivalent.*

COROLLARY 2. *There exist uncountably many inequivalent Antoine decompositions of E^3.*

REFERENCES

1. M. L. Antoine, *Sur la possibilité d'étendre l'homéomorphie de deux figures à leur voisinages*, C.R. Acad. Sci. Paris **171** (1920), 661–663.

2. ———, *Sur l'homéomorphie de deux figures et de leurs voisinages*, J. Math. Pures Appl. **86** (1921), 221–325.

3. S. Armentrout, *Decompositions of E^3 with a compact 0-dimensional set of non-degenerate elements*, Trans. Amer. Math. Soc. **123** (1966), 165–177.

4. ———, "Monotone decompositions of E^3" in *Wisconsin topology conference*, Ann. of Math. Studies No. 60, Princeton, N. J., 1966, pp. 1–25.

5. S. Armentrout and R. H. Bing, *A toroidal decomposition of E^3*, Fund. Math. **60** (1967), 81–87.

6. S. Armentrout, L. L. Lininger and D. V. Meyer, "Equivalent decompositions of E^3" in *Wisconsin topology conference*, Ann. of Math. Studies, No. 60, Princeton, N. J., 1966, pp. 27–31.

7. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362.

8. ———, *Point-like decompositions of E^3*, Fund. Math. **50** (1962), 437–453.

9. A. Kirkor, *Wild 0-dimensional sets and the fundamental group*, Fund. Math. **45** (1958), 228–236.

10. H. W. Lambert and R. B. Sher, *Point-like 0-dimensional decompositions of S^3*, Pacific J. Math. **24** (1968), 511–518.

11. R. B. Sher, *Toroidal decompositions of E^3*, Fund. Math. **61** (1968), 225–241.

THE UNIVERSITY OF GEORGIA