ON A POSITIVE TRIGONOMETRIC SUM

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We give a new proof of the following theorem of P. Turán [2]. See [1] for another proof, shorter than Turán's but longer than this proof.

THEOREM. *Let* $\sum_{n=1}^{N} a_n \sin(2n-1)\theta \geqq 0,\ 0 \leqq \theta \leqq \pi$. *Then*

$$\sum_{n=1}^{N} \frac{a_n \sin n\phi}{n} > 0, \qquad 0 < \phi < \pi,$$

unless all $a_n = 0$.

A simple computation shows that

$$\frac{d}{dy} \frac{\sin \alpha y}{\alpha(\sin y)^{\alpha}} = -\frac{\sin(\alpha - 1)y}{(\sin y)^{\alpha+1}}.$$

Letting $\alpha = 2n$ and $y = \phi/2$ we see that

$$\frac{\sin n\phi}{n} = 2\int_{\phi/2}^{\pi/2} \left(\frac{\sin \phi/2}{\sin \theta}\right)^{2n} \frac{\sin(2n-1)\theta}{\sin \theta} d\theta.$$

Thus

$$\sum_{n=1}^{N} \frac{a_n \sin n\phi}{n} = 2\int_{\phi/2}^{\pi/2} \sum_{n=1}^{N} a_n \left(\frac{\sin \phi/2}{\sin \theta}\right)^{2n} \sin(2n-1)\theta \frac{d\theta}{\sin \theta}.$$

But $\sum_{n=1}^{N} a_n r^{2n-1} \sin(2n-1)\theta > 0,\ 0 < r < 1$, if $\sum_{n=1}^{N} a_n \sin(2n-1)\theta \geqq 0$ and not all a_n are zero and this completes the proof.

REFERENCES

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2. P. Turán, *On a trigonometric sum*, Ann. Polon. Math. 25 (1953), 155–161.

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