HOMOTOPIC ARCS ARE ISOTOPIC

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The purpose of this note is to show that if α and β are flat arcs in an n-manifold M ($n \geqq 3$) and α and β are homotopic (with fixed endpoints), then there is an isotopy of M onto itself, leaving the endpoints of α and β fixed, which carries α onto β. This result is actually an application of an unstated theorem in §6 of [2] and is of primary interest when $n=3$, as theorems of this kind are well known for larger values of n.

This same sort of technique has been used by James Kister to show that each arc in an n-manifold is isotopic, with fixed endpoints, to a flat arc. Of course, in this case, the isotopy is not ambient.

An *n-manifold* is a topological space which may be covered by open sets, each of which is homeomorphic to E^n, Euclidean n-dimensional space. An arc in an n-manifold M is *flat* if there exists a closed neighborhood A of α and a homeomorphism h of A onto D^n, the unit ball in E^n, which carries α onto a straight line interval. A *homotopy* of a space X in a space Y is a map $f: X \times I \rightarrow Y$; we shall sometimes write $f_t(x)$ for $f(x, t)$ or use another closed real interval to replace $I=[0, 1]$. We say that f is *fixed* on a subset S of X if $f_t | S = f_{t'} | S$ whenever $t, t' \in I$. If f_t is a homeomorphism for each t, we call f an *isotopy*, and if each f_t is also surjective and f_0 is the identity, then f is an *ambient isotopy of X*. A *path* in a space Z is a map of I into Z. The paths ω and ω' are *path homotopic* in Z if there is a homotopy g of I in Z, fixed on $\{0, 1\}$ such that $\omega = g_0$ and $\omega' = g_1$. If two arcs are images of I by homeomorphisms which are path homotopic, we shall also call the arcs *path homotopic*. It may easily be seen that two arcs α and β are path homotopic if and only if they have common endpoints and there exists a path λ such that λ maps $[0, \frac{1}{2}]$ homeomorphically onto α, $[\frac{1}{2}, 1]$ homeomorphically onto β, and λ is path homotopic to a constant path.

THEOREM. *Suppose that $n \geqq 3$, M is an n-manifold, and α and β are path homotopic flat arcs in M with common endpoints p and q. Then there exists an ambient isotopy h_t $(0 \leqq t \leqq 1)$ of M, fixed on p and q, such that $h_1(\alpha) = \beta$. Furthermore, if K is a closed set such that α and β are path homotopic in $M - K$ then we may require that h_t be fixed on K.*

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PROOF. Let M, α, β, p, q, and K be as in the hypothesis and let λ be a path in M such that $\lambda|[0, \frac{1}{2}]$ and $\lambda|[\frac{1}{2}, 1]$ are homeomorphisms onto α and β respectively, $\lambda(0)=\lambda(1)=p$, and λ is path homotopic in $M-K$ to the constant path at p.

Now let A be a closed neighborhood of α in $M-K$ such that there exists a homeomorphism carrying A onto D^n and α onto a straight line interval, and let B be a similar neighborhood of β in $M-K$. Let C be an n-cell in $M-K$ containing q in its interior. Let u and v be numbers such that $0<u<\frac{1}{2}<v<1$ and $\lambda([u, v])\subset \text{int } C$.

We next describe an ambient isotopy of M which carries α onto β, leaves $K\cup\{q\}$ fixed, but which moves p. We will later modify this isotopy so as to leave p fixed.

In D^n, any straight line interval in the interior may be shrunk through itself toward an endpoint by an ambient isotopy, fixed on that endpoint and on Bd D^n. Using this fact we may define an isotopy g_t $(0\leq t\leq u)$ of M onto itself such that:

$$g_0 = \text{identity}, \qquad g_u(\alpha) \subset \text{int } C,$$
$$g_t | M - A = \text{identity}, \qquad g_t(q) = q$$

and

$$g_t(p) = \lambda(t).$$

Now let α' be $g_u(\alpha)$ and β' be $\lambda([\frac{1}{2}, v])$. Then α' and β' are flat arcs interior to C and sharing the endpoint q. Thus there is an ambient isotopy of C, fixed on Bd $C\cup\{q\}$, which takes α' to β'. This may be used to extend g_t to the interval $0\leq t\leq v$ so that it satisfies

$$g_v(\alpha) = \beta', \qquad g_t | M - (A \cup C) = \text{identity},$$

and

$$g_t(q) = q.$$

Notice that $g_v(p)=\lambda(v)$.

Finally, by a reversal of the first type of isotopy, we extend g_t to an isotopy of M onto M for $0\leq t\leq 1$, such that

$$g_1(\alpha) = \beta, \qquad g_t | M - (A \cup B \cup C) = \text{identity}, \quad g_t(q) = q$$

and

$$g_t(p) = \lambda(t) \qquad \text{if } 0 \leq t \leq u \quad \text{or} \quad v \leq t \leq 1.$$

Now let $\omega(t)$ be the path $g_t(p)$. Notice that ω and λ differ only for $u\leq t\leq v$, and for these values of t, $\omega(t)$ and $\lambda(t)$ both lie in the n-cell C. Therefore ω and λ are path homotopic in $M-K$ and hence ω is path homotopic in $M-K$ to the constant path at p. Now since $n\geqq 3$ and $q\notin\omega(I)$, ω is also path homotopic to the constant path at p in $M-(K\cup\{q\})$. That is, there exists a homotopy F: $I\times I\rightarrow M-(K\cup\{q\})$ such that

$$F(s, 0) = \omega(s) = g_s(p)$$

and

$$F(s, 1) = F(0, t) = F(1, t) = p.$$

We now apply the technique in [2]. Let $\mathcal{H}$ be the space of all homeomorphisms of M onto M which leave $K\cup\{q\}$ pointwise fixed, where $\mathcal{H}$ has the compact open topology. Define π: $\mathcal{H}\rightarrow M-(K\cup\{q\})$ by $\pi(h)=h(p)$. It is well known that the triple $(\mathcal{H}, \pi, M-(K\cup\{q\}))$ is a fiber bundle.

Now an ambient isotopy of M which is fixed on $K\cup\{q\}$ is simply a path in $\mathcal{H}$ beginning at the identity. In particular the path ϕ in $\mathcal{H}$ defined by $\phi(t)=g_t$ $(0\leqq t\leqq 1)$ satisfies $\pi\circ\phi=F_0$. Now by the homotopy lifting property, there is a homotopy G: $I\times I\rightarrow\mathcal{H}$ such that $\pi\circ G=F$ and $G_0=\phi$. Since π then maps each of $G(s, 1)$, $G(0, t)$, and $G(1, t)$ into p, these are all members of $\mathcal{H}$ which leave p fixed. Now define h_t $(0\leqq t\leqq 1)$ as follows:

$$\begin{aligned} h_t &= G(0, 3t), & 0 &\leqq t \leqq 1/3, \\ &= G(3t-1, 1), & 1/3 &\leqq t \leqq 2/3, \\ &= G(1, 3-3t), & 2/3 &\leqq t \leqq 1. \end{aligned}$$

Then h is an ambient isotopy of M, $h_0=g_0=$identity, $h_1(\alpha)=g_1(\alpha)=\beta$, and h is fixed on $K\cup\{p, q\}$. This completes the proof.

We remark that if M is simply connected, the conclusion of the theorem holds for any two flat arcs with common endpoints. Also, a slight modification of the argument yields a similar theorem if M is a manifold with boundary, provided that one of the common endpoints of α and β is interior to M. Finally, we remark that in [1] Feustel proves a theorem of this type for nonambient isotopies of arcs in 2-manifolds with boundary, where the endpoints of the arcs are on the boundary, and also gives an example to show that these theorems are false for 2-manifolds.

References

1. C. D. Feustel, *Homotopic arcs are isotopic*, Proc. Amer. Math. Soc. **17** (1966), 891–896.

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