THE UNIQUENESS OF A CERTAIN LINE INVOLUTION

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1. **Introduction.** In 1933 in a paper in the *Bulletin of the American Mathematical Society* J. M. Clarkson [1] discussed the following line involution: In a projective space of three dimensions over the field of complex numbers, let $\bar{\pi}_1$ and $\bar{\pi}_2$ be two planes intersecting in a line μ. In $\bar{\pi}_1$ let H_1 be a harmonic homology whose axis, d_1, is distinct from μ and whose center, C_1, does not lie on μ. In $\bar{\pi}_2$ let H_2 be a harmonic homology whose axis, d_2, is distinct from μ and whose center, C_2, does not lie on μ. The image of a general line, l, meeting $\bar{\pi}_1$ and $\bar{\pi}_2$ in points L_1 and L_2, respectively, is the line l' determined by the points L_1' and L_2' which are the images of L_1 and L_2 under H_1 and H_2, respectively. Among other things, Clarkson showed that the order, m, of this involution is two, that the order, i, of its complex of invariant lines is zero, and that the image of a general plane field of lines is a bilinear congruence with distinct directrices. Also, although Clarkson did not note this fact, it follows readily from his work, or from the formula [2] $k=m-2i+1$ that the order, k, of the complex of singular lines, or lines which meet their images, is three. Furthermore, in this involution the exceptional lines, or lines which do not have a unique image, are not E-singular, that is they do not meet the exceptional line in their respective image-families.

The particular involution investigated by Clarkson is not the only one with the properties that $m=2$, $i=0$, $k=3$ and that the image of a general plane field of lines is a bilinear congruence with distinct directrices. In fact the line involution defined by any two harmonic homologies, H_1: (d_1, C_1) and H_2: (d_2, C_2), in distinct planes has these properties except in the following cases:

(i) d_1, d_2, and μ coincide,

(ii) d_1 and d_2 intersect on μ, and C_1 and C_2 coincide on μ.

In (i), the involution reduces to the line involution defined by the skew homology whose axes are μ and the line determined by C_1 and C_2. In (ii) the involution is the line involution defined by the central homology whose plane is the plane determined by d_1 and d_2 and whose center is the point on μ in which C_1 and C_2 coincide. In what follows, we shall refer to any line involution defined by two harmonic homologies in distinct planes as Clarkson's involution except when the homologies are specialized as in (i) or (ii).

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The purpose of this paper is to prove that Clarkson's involution is unique; more specifically we shall prove the following theorem:

THEOREM 1. *Clarkson's involution is the only line involution with the following properties*:

(i) $m=2$, $i=0$, $k=3$,

(ii) *The image of a general plane field is a bilinear congruence with distinct directrices*,

(iii) *The exceptional lines are not E-singular.*

In our proof we shall work primarily on the nonsingular hyperquadric, V_4^2, in S_5 onto which the lines of S_3 are mapped by the well-known device of interpreting the Plücker coordinates of the lines of S_3 as point coordinates in S_5. In thus studying line involutions in S_3 as point involutions of V_4^2 into itself, we shall for convenience refer to the planes of V_4^2 which are the images of plane fields of lines in S_3 as f-planes, and we shall refer to the planes of V_4^2 which are the images of bundles of lines in S_3 as b-planes.

2. **The mapping of a general f-plane of V_4^2.** Under the hypotheses of Theorem 1, the image of a general line of V_4^2 is a nonsingular conic, and the image of a general f-plane, π_f, is a nonsingular quadric surface, Q. Since two conics on Q intersect in two points, whereas two lines of π_f have a single point in common, it follows that the lines of π_f must be transformed into a family of conics on Q having a fixed point, P, in common. Moreover, since P is a simple point on each of these conics, it must have a single pre-image on each line of π_f. In other words, the total pre-image of P in π_f must be a line, p.

The line p, though not an ordinary line, must also have a conic image on Q. Hence it must contain either one exceptional point with a conic image or two exceptional points each with a line of images. The former possibility can be rejected at once, because a line of π_f through such a point could not have a proper image curve, as it must since its points are general points of π_f. Hence p contains two points, G_1 and G_2, each of which is transformed into a line on Q. Moreover, since these lines each contain P, they are, in fact, the generators, g_1 and g_2, of Q which pass through P.

Since the image of G_i is the line g_i, it follows that the image of a general line, l, through G_i in π_f is a composite conic having g_i as one component. The other component, which is the proper image of l, meets g_i and must therefore be a generator of the regulus, R_j, to which g_j belongs. In other words, if R_1 and R_2 are the reguli of Q which contain g_1 and g_2, respectively, then the image of the pencil (π_f, G_1) is the regulus R_2 and the image of the pencil (π_f, G_2) is the regulus R_1. From this it is clear that the proper images of the conics of π_f which contain G_1 and G_2 comprise the totality of conics on Q.

3. **The E-point locus on V_4^2.** Since the proper image of a general generator of the image quadric, Q, of a general f-plane, π_f, is a line in π_f, it follows that on each generator of Q there must be one exceptional point, E, with a line of images. The locus of these E-points on Q must therefore be either a proper conic or a pair of generators, one from each regulus. Moreover, since a general conic, C, through the point P on Q has a line in π_f as its proper image, it follows that the two intersections of C and the E-conic on Q must be distinct from P. Hence the pre-image in π_f of the E-conic on Q cannot be a single line or the pair of points G_1 and G_2, but must be either a proper conic or a pair of lines, one on G_1 and one on G_2.

Since a general f-plane contains two and only two exceptional points, namely G_1 and G_2, it is clear that on V_4^2 the locus of E-points can be neither a complex nor a curve. Therefore the locus of E-points must be a surface, and the following possibilities exist:

(i) The E-locus contains ∞^1 E-conics each lying on the image quadrics of ∞^2 f-planes.

(ii) The E-locus contains ∞^2 E-conics each lying on the image quadrics of ∞^1 f-planes.

(iii) The E-locus contains ∞^3 E-conics each lying on the image quadric of a single f-plane.

The possibility (i) can clearly be rejected, for the pre-image of any particular E-conic, C, is a ruled surface necessarily lying in the S_4 which is tangent to V_4^2 at the point of intersections of any two of the f-planes whose image quadrics contain C. Hence every f-plane whose image contains C must be in this S_4, which is impossible since a tangent S_4 contains only ∞^1 f-planes of V_4^2.

In (iii), if the ∞^3 E-conics are proper they necessarily lie on a non-composite quadric surface, while if they are composite their components necessarily lie, respectively, in two planes of V_4^2 with a line in common. Let π_f and π_f' be two f-planes intersecting in a general point O, and let Q and Q' be their respective image quadrics. In addition, let C and C' be the E-conics on Q and Q', respectively, and let A and B be the points common to C and C'. Then if O' is the image of O, the plane π determined by O', A, B lies in both the S_3 containing Q and the S_3 containing Q'. Thus π has the same conic C'' in common with both Q and Q' and, perforce, every point of C'' has a pre-image in both π_f and π_f'. Hence C'', which is surely distinct from the E-

conics C and C' since it contains the image of the general point O, is a second conic of E-points on both Q and Q', which is impossible. Thus (ii) remains the only possibility, that is, the E-locus contains ∞^2 E-conics each of which lies on the image quadrics of ∞^1 f-planes.

Now the ∞^1 f-planes which are the pre-images of the ∞^1 quadrics which have a particular E-conic in common can have no free intersection. Hence they must consist of all the f-planes through some point F; that is, they must comprise the totality of f-planes in the special linear complex, Γ_F, which is cut from V_4^2 by the tangent S_4 at F. Moreover F must be the pre-image of an E-point, for otherwise the image quadrics of the planes of Γ_F would all have an E-conic as well as the image of F in common, and hence would coincide. If the E-conic under consideration is proper, its pre-image curve in each f-plane of Γ_F is a proper conic containing F. Thus the pre-image locus of the E-conic, being a ruled surface in Γ_F with conic directrices intersecting in a self-corresponding point, must be a cubic surface. Therefore the pre-image locus of the surface of E-points must be a V_3^3 since it is intersected by S_4's in cubic surfaces. But if this is the case, then the E-point pre-image locus in a general f-plane must be a cubic curve, whereas it actually is only a conic. Thus the E-conic on the image quadric of a general f-plane cannot be proper but must, instead, be composite with a composite pre-image in that plane consisting of one line on each of the exceptional points G_1 and G_2 in that plane. Moreover, since each pre-image line of a component of any E-conic contains an E-point, it is clear that the E-locus is, in fact, contained in the E-point pre-image locus.

Now since the E-locus contains ∞^2 composite conics, it must be either a nonsingular quadric surface or a pair of planes. It cannot be the former, however, since each f-plane must contain two E-points. Hence the E-locus must consist of a pair of f-planes, π_1 and π_2, intersecting in a point M, and the ∞^2 composite E-conics are simply the conics consisting of an arbitrary line of the pencil (π_1, M) and an arbitrary line of the pencil (π_2, M). Moreover each of the linear complexes Γ_1 and Γ_2, comprising the pre-image of the E-surface, contains one of the plane components of the E-surface and hence is special.

Finally, since it is part of our hypothesis that the E-lines of our involution are not E-singular, it follows that π_1 does not lie in Γ_1 and π_2 does not lie in Γ_2. In other words, the E-point on the pre-image line of an E-point in π_i lies in π_j.

4. **The complex of singular elements.** Since the image of a general f-plane is a nonsingular quadric surface Q, it is clear that a general

f-plane, π_f, has either a single point or a line in common with its image. If the intersection is a line, its points cannot be invariant for this would imply the existence of a complex of invariant lines, whereas $i=0$. On the other hand, the points of such a line cannot be exceptional, for a general f-plane contains only two E-points. Hence the line must be transformed into itself in such a way that precisely two of its points, I and I', are left invariant. Moreover, since the line II' is a generator of Q which is its own image, it follows that it must pass through one of the two exceptional points in π_f, say G_1. The lines determined by the other exceptional point, G_2, and the invariant points I and I' are then transformed into generators, r_1 and r_1', of the regulus R_1 on Q. Now the component, e_2, of the E-conic on Q which belongs to R_2 meets r_1 and r_1' in A and A', say, and has as its pre-image in π_f a line, $\bar{e}_2$, on G_1. Furthermore, since E-points are not E-singular, it is clear that the E-point on $\bar{e}_2$, namely G_1, does not lie in the E-plane which contains e_2, so that it is G_2 which lies in the same E-plane with e_2. Thus G_2A and G_2A' are lines of V_4^2, and hence the planes G_2IA and $G_2I'A'$ lie entirely on V_4^2. Therefore the lines G_2I and G_2I' are lines of singular points as, of course, is II'.

The cubic complex of singular elements meets the quadric Q in a composite sextic consisting of r_1, r_1', II_1' and a residual component of order three. Moreover, even if this is a composite cubic containing the generators whose pre-images in π_f are G_1 and G_2 there is at least a linear component whose pre-image in π_f is different from G_2I, G_2I', and II'. But this is impossible since the intersection of π_f and the complex of singular elements is only a cubic curve. Hence π_f and its image quadric cannot have a line in common, and hence must intersect in a single point, which is necessarily invariant.

Let I be the invariant point common to a general f-plane π_f and its image quadric, Q. Then the lines G_1I and G_2I are transformed respectively into the generators, r_2 and r_1, of Q which contain I. Moreover, as before, the points of G_1I, G_2I, r_1, and r_2 are all singular, and the plane, α_1, determined by G_1I and r_2 and the plane, α_2, determined by G_2I and r_1 are b-planes on V_4^2.

Now in α_1, since G_1I is transformed into r_2 in such a way that I is self-corresponding, it follows that the transformation is a perspectivity with center, S, say. Let l be the line joining a general point, A, on G_1I to its image, A', on r_2. Since the image of l must contain A and A', it is clear that if the image of l is a line, then every point of l, and hence every point of α_1, is singular with image in α_1. Similarly, if the image of l is a conic lying in a b-plane, π_b, or is an entire b-plane, π_b, then π_b must coincide with α_1, and again every point of α_1 is singular with image in α_1. On the other hand, if the image of a line l joining a general point on G_1I to its image on r_2 is a conic in an f-plane, then the f-planes associated with two such lines would have in common both S and its image S' and hence would have to coincide, which is impossible. Finally, if l is transformed into an entire f-plane or some other surface on V_4^2, then each of the ∞^3 f-planes on V_4^2 must meet the image of l in at least one point, and hence the image quadric of each f-plane must have a point in common with l. Furthermore, in α_1 there are ∞^1 lines joining a point on G_1I to its image on r_2. Hence, given a general f-plane, its image quadric, Q, must have a curve in common with α_1 and, since Q is not composite, this curve must be a straight line. Now since there exist ∞^2 α_1-planes, each of which is a b-plane on V_4^2, it follows that Q either contains ∞^2 lines or else contains ∞^1 lines each of which is common to ∞^1 b-planes. Since each of these is impossible, we conclude that every point of α_1 is singular with image in α_1. Hence the cubic complex of singular elements is composite, consisting of three linear complexes, at least two of which, Γ_{α_1} and Γ_{α_2}, are special. Moreover, since each plane, α_1, contains the corresponding E-point, G_1, and since in each f-plane the point G_1 is in π_1, it follows that Γ_{α_1} contains the E-plane π_1. Similarly, Γ_{α_2} contains the E-plane π_2. Hence the vertex, D_1, of Γ_{α_1} lies in π_1 and the vertex, D_2, of Γ_{α_2}, lies in π_2.

The vertices D_1 and D_2 correspond to two lines, d_1 and d_2, in S_3 lying respectively in the planes, $\bar{\pi}_1$ and $\bar{\pi}_2$, which are represented on V_4^2 by the E-planes π_1 and π_2. Thus the complex Γ_{α_1} consists of all lines meeting d_1, and the complex Γ_{α_2} consists of all lines meeting d_2. Moreover, since the b-planes of both Γ_{α_1} and Γ_{α_2} are transformed into themselves, it follows that in S_3 a general bundle with vertex on either d_1 or d_2 is transformed into itself. Hence all lines in S_3 which meet both d_1 and d_2 are invariant, and on V_4^2 there is a nonsingular quadric surface of invariant points.

5. **Conclusion.** On V_4^2 let π_b be an arbitrary b-plane meeting π_1 in a line l. Let A be a general point in π_b, let its image be A', let λ be any line in π_b which contains A, and let π_f be the f-plane which contains λ. Then the intersection of λ and l is the point G_1 in π_f. Hence λ is transformed into a generator, r_2, of the R_2 regulus on Q, and therefore r_2 contains an E-point on the e_1-component of the E-conic on Q. Thus the image of λ, which of course passes through A', meets π_1. Now the lines of V_4^2 which pass through A' and meet π_1 form a pencil lying in the b-plane which contains A' and meets π_1 in a line. Hence, considering all the lines of the pencil (π_b, A) it is clear that the proper image of

π_b is a second b-plane intersecting π_1 in a line. Thus in S_3, any bundle whose vertex is a point of $\bar{\pi}_1$ has for its proper image another bundle whose vertex is also in $\bar{\pi}_1$. A similar observation, of course, holds for any bundle whose vertex is a point of $\bar{\pi}_2$.

We have thus shown that associated with any line involution possessing the properties described in the hypothesis of Theorem 1, there are two planes, $\bar{\pi}_1$ and $\bar{\pi}_2$, such that the line involution induces point involutions in each of these planes. Furthermore, since the order of the line involution is two, it is clear that the associated point involutions in $\bar{\pi}_1$ and $\bar{\pi}_2$ must be linear. Hence, since the points of the directrices, d_1 and d_2, of the two special linear complexes of singular lines are invariant under the respective point involutions, it follows that these point involutions must be harmonic homologies with d_1 and d_2 as their axes. The piercing points on $\bar{\pi}_1$ and $\bar{\pi}_2$ of the image, l', of a general line, l, in S_3 are thus the images of the intercepts of l under harmonic homologies in $\bar{\pi}_1$ and $\bar{\pi}_2$, and the proof of Theorem 1 is complete.

References

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