n-PERSON GAMES WITH ONLY 1, $n-1$, AND n-PERSON COALITIONS

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A considerable amount of work has been done on n-person games in which all coalitions with less than $n-1$ players are totally defeated. Maschler [3] has studied the structure of their bargaining sets, while Lucas, in two separate papers [1], [2] has given a symmetric solution to such games and studied the behavior of solutions for such games in partition function form. In this paper, we propose to give all discriminatory solutions to such games.

We consider here n-person games, v, satisfying

(1) $$v(N) = 1,$$

(2) $$0 \leqq v(N - \{j\}) \leqq 1 \quad \text{for all } j \in N,$$

(3) $$v(S) = 0 \quad \text{for } |S| < n - 1,$$

where $|S|$ denotes the number of players in the coalition S, and $N = \{1, 2, \cdots, n\}$. We shall assume throughout that $n \geqq 3$. We shall write $d_j = 1 - v(N - \{j\})$.

We now seek discriminatory solutions for such games. The following definition and two theorems, given without proof, are from [4].

DEFINITION. Let v be an n-person game, let S be a coalition with $s = |S|$, and q a number. Then $\bar{v}_{S,q}$ is the s-person game defined by

$$\bar{v}(T) = v(T), \quad \text{if } T \subset S \text{ but } T \neq S, \qquad \bar{v}(S) = q.$$

THEOREM 1. *Let v be an n-person game, and let V be a solution to v which discriminates the members of $N-S$, giving them the amounts α_j. Let V^* be obtained from V by taking the S-components of the elements of V. Then V^* is a solution to $\bar{v}_{S,q}$, where*

(4) $$q = v(N) - \sum_{N-S} \alpha_j,$$

THEOREM 2. *Let v be an n-person game, let α_j, for $j \in N-S$, be constants, and q defined by* (4). *Let V^* be a solution to $\bar{v}_{S,q}$, and suppose V is obtained from V^* by adjoining the components (α_j) to each element of V^*. Then V is internally stable, and, moreover, a necessary and sufficient condition for the set V to dominate all imputations x such that*

(5) $$\sum_{N-S} (x_j - \alpha_j) > 0$$

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is that either $v(S) \geqq q$ or the core of the game $\bar{v}_{S,q}$ have no interior points.

Suppose, then, that the game v satisfies (1)–(3). We are looking for solutions which discriminate at least one player. It is easy to see that there are no solutions which discriminate more that one player, since if $N-S$ has more than one player, then by (3) $v(S)=0$. For any positive q, all the imputations of $\bar{v}_{S,q}$ would belong to the core. By Theorem 2, we could have such a solution only for $q=0$. But this would make V^*, and hence also V, have only one point, and we know that all solutions to essential games have at least 3 points.

Let, then, $i \in N$ be a player. We look for solutions which will give player i the constant amount α. Let $S=N-\{i\}$.

None of the proper subsets of S is effective, and so, for $q>0$, the core of $\bar{v}_{S,q}$ has a nonempty interior. By Theorem 2, then, we must have

$$\alpha = 1 - q \geqq 1 - v(N - \{i\}) = d_i. \tag{6}$$

Since all the imputations in $\bar{v}_{S,q}$ are in the core, it follows that the set of all imputations is the unique solution to $\bar{v}_{S,q}$, and so

$$V = \{x \mid x_j \geqq 0 \text{ for all } j, \textstyle\sum x_j = 1, x_i = \alpha\}. \tag{7}$$

By Theorem 2, we know that V is internally stable and dominates all imputations x with $x_i > \alpha$. Since all the imputations with $x_i = \alpha$ belong to V, we need be concerned only with the imputations such that $x_i < \alpha$.

Suppose, then, that $x_i < \alpha$. Can x be dominated by some $y \in V$? If so, this domination must be through an $(n-1)$-person coalition, $N-\{j\}$, since no other coalitions are effective for domination. Clearly, we must have $j \neq i$, since $x_i < \alpha = y_i$, and this means that some member of $N-\{i\}$ gets more in x than in y. We find, then, that for domination, y must satisfy

$$y_k > x_k \quad \text{for all } k \neq j, \tag{8}$$

$$\sum_{k \neq j} y_k \leqq v(N - \{j\}). \tag{9}$$

Condition (9) can be restated in the form

$$y_j \geqq d_j. \tag{10}$$

Now we know $y_i = \alpha$. It follows that (8), (10) can be satisfied if and only if

$$\sum_{i \neq k \neq j} x_k + d_j < q. \tag{11}$$

For, in fact, if (11) is satisfied, we have

$$\sum_{i\neq k\neq j} x_k = q - d_j - \epsilon$$

where $\epsilon > 0$, and we can set

$$\begin{aligned} y_k &= \alpha \quad \text{for } k = i, \\ &= d_j \quad \text{for } k = j, \\ &= x_k + \epsilon/(n-2) \quad \text{for } i \neq k \neq j. \end{aligned}$$

It is easily checked that y is an imputation, satisfies (8), (10), and belongs to V. On the other hand, it is easy to see that if any $y \in V$ satisfies (8), (10), then (11) must hold since $q = 1 - \alpha$. We thus obtain the following result:

LEMMA 1. *A necessary and sufficient condition for the imputation x, with $x_i < \alpha$, to be undominated by the set V, is that, for every $j \neq i$,*

$$x_i + x_j \leqq \alpha + d_j. \tag{12}$$

PROOF. Since $\sum x_k = 1$, and $\alpha + q = 1$, it follows that (11) and (12) are contradictory opposites. Thus if (12) holds, x is undominated by V through $N - \{j\}$, and conversely. But there are no other coalitions to be considered.

We wish to know, now, whether there is any imputation x which satisfies (12) for all $j \neq i$, i.e., whether the inequalities (12) are consistent with the usual inequalities $\sum_N x_k = 1$, $x_k \geqq 0$. If so, then V is not a solution; if they are inconsistent, then V is a solution. We have then

THEOREM 3. *The set V, satisfying* (6)–(7), *is a solution if and only if either $\alpha = 0$ or*

$$\alpha < \left(1 - \sum_{j\neq i} d_j\right) \Big/ (n-1). \tag{13}$$

PROOF. Suppose first that $\alpha = 0$. Then there are no x with $x_i < \alpha$, and so by Theorem 2, V is a solution. Suppose (13) holds. Then, for any imputation x,

$$(n-1)\alpha + \sum_{j\neq i} d_j < 1 = x_i + \sum_{j\neq i} x_j,$$

$$(n-1)\alpha + \sum_{j\neq i} d_j < (n-1)x_i + \sum_{j\neq i} x_j$$

and so, for at least one $j \neq i$, $\alpha + d_j < x_i + x_j$ so that (12) cannot hold. If $x_i < \alpha$, this means by Lemma 1 that x is dominated by some $y \in V$. By Theorem 2, this means V is a solution.

Conversely, suppose $\alpha > 0$ and (13) does not hold. We have, then,

$$(n-1)\alpha + \sum_{j \neq i} d_j = 1 + \delta$$

where $\delta \geqq 0$. In this case, we take x, defined by

$$x_i = 0, \qquad x_j = (\alpha + d_j)/(1+\delta) \quad \text{for } j \neq i.$$

Clearly, x is an imputation which satisfies the conditions (12) and so is undominated by V. Since $x_i = 0 < \alpha$, then $x \notin V$, and so V is not a solution.

Substituting (6) in (13), we obtain a final

THEOREM 4. *A game satisfying* (1)–(3) *will have a discriminatory solution, discriminating player i, if and only if either*

$$d_i = 0 \tag{14}$$

or

$$(n-1)d_i + \sum_{j \neq i} d_j < 1. \tag{15}$$

PROOF. If (14) holds, then by Theorem 3, the set V with $\alpha = 0$ is a solution. If (15) holds, then there is some α satisfying (6) and (13) and this will give a solution. If neither holds, then conditions (6) and (13) cannot be satisfied and there is no such solution.

We have, thus, discovered all the discriminatory solutions to games such as these. As we pointed out above, Lucas in [1] gives the symmetric solutions. These are not, however, the only solutions to such games. In many cases, part of the hyperplane which forms the discriminatory solution can be replaced by the $n-1$ dimensional analogue of the bargaining curves which appear in the solutions of general-sum 3-person games. In other games, the core is nonempty, and a solution will consist of the core, together with several of these "bargaining surfaces." There may be even other types of solutions; further investigation is necessary to say more.

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