UMBILICAL HYPERSURFACES IN AFFINELY CONNECTED SPACES

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In this paper we present a definition of an umbilical hypersurface in an m-dimensional space M of affine connection. Some properties of such hypersurfaces are found when M is assumed to be projectively flat.

In general we follow the notation in [**3**] but adopt the positioning of indices used in [**4**]. Thus the usual summation convention holds. Greek indices range over the dimension of M, Latin indices take the values $1, 2, \cdots, n=m-1$. Let e_α be a base field and $\bar{w}^\alpha$ the dual 1-forms associated with it. We assume the connection on M to be torsion-free which implies that [**3**, p. 62]

$$0 = d\bar{w}^\alpha + \bar{w}^\alpha{}_\lambda \wedge \bar{w}^\lambda. \tag{1}$$

We also make use of the second Cartan structural equation [**3**, p. 63]

$$\bar{R}^\alpha{}_\beta = d\bar{w}^\alpha{}_\beta + \bar{w}^\alpha{}_\lambda \wedge \bar{w}^\lambda{}_\beta. \tag{2}$$

For any vector V on M we define a 1-form $\bar{R}_\beta(V)$ as follows:

$$(m^2 - 1)\bar{R}_\beta(V) = (m + 1)\bar{R}^\lambda{}_\beta(V, e_\lambda) - \bar{R}^\lambda{}_\lambda(V, e_\beta). \tag{3}$$

On account of the first Bianchi identity which is valid because of (1) [**4**, p. 51] we find that

$$(m + 1)\bar{w}^\lambda \wedge \bar{R}_\lambda + \bar{R}^\lambda{}_\lambda = 0. \tag{4}$$

The Weyl tensor [**1**, p. 88], [**5**, p. 289] is expressed by the 2-form

$$\bar{P}^\alpha{}_\beta = \bar{R}^\alpha{}_\beta + \bar{w}^\lambda \wedge \bar{R}_\lambda \delta^\alpha{}_\beta + \bar{w}^\alpha \wedge \bar{R}_\beta. \tag{5}$$

Here it might be helpful to consult [**6**] in comparing notations. For $m>2$, the space M is called projectively flat if $\bar{P}^\alpha{}_\beta=0$. This implies

$$d\bar{R}_\beta + \bar{R}_\lambda \wedge \bar{w}^\lambda{}_\beta = 0. \tag{6}$$

For $m=2$, $\bar{P}^\alpha{}_\beta$ is identically zero and a projectively flat space is then defined by (6).

In dealing with a hypersurface N we let the vectors e_i be tangent to N and let e_m be the local "normal" [**1**, pp. 138, 155]. Restricting (2) to vectors on N we have [**3**, p. 82]

$$\bar{R}^i{}_j = R^i{}_j + w^i{}_m \wedge w^m{}_j. \tag{7}$$

Received by the editors August 18, 1967.

DEFINITION. A hypersurface N is locally umbilical if for vectors on N

$$w^i{}_m = w^i. \tag{8}$$

THEOREM. *An umbilical hypersurface N of a projectively flat space M is itself projectively flat.*

PROOF. Using (7), (8), (1) we infer $\overline{R}^h{}_h = R^h{}_h$. Now we rewrite $\overline{P}^m{}_m = 0$ by means of (5) and (4) and get

$$m\overline{R}^\lambda{}_\lambda = (m+1)R^h{}_h. \tag{9}$$

Also, expressing $\overline{P}^m{}_j(X, e_m) = 0$ with the aid of (5) leads to

$$\overline{R}^m{}_j(X, e_m) = \overline{R}_j(X) \tag{10}$$

for a vector X belonging to N. Relations (7) and (8) yield

$$\overline{R}^h{}_j(X, e_h) = R^h{}_j(X, e_h) - (n-1)w^m{}_j(X). \tag{11}$$

If we apply the definition (3) to N we conclude from (9), (10), (11) that

$$\overline{R}_j = R_j - w^m{}_j. \tag{12}$$

Combination of (5), (7), (8), (9), (12) gives $\overline{P}^i{}_j = P^i{}_j$. This tells us that N is projectively flat if n is at least 3.

For $n=2$ we make use of (5) to change $\overline{P}^h{}_m(X, e_h) = 0$ to $\overline{R}^h{}_m(X, e_h) - (n-1)\overline{R}_m(X) = 0$. On the other hand, (2) in conjunction with (8) and (1) becomes $\overline{R}^h{}_m(X, e_h) + (n-1)w^m{}_m(X) = 0$. By comparison,

$$w^m{}_m = -\overline{R}_m. \tag{13}$$

Moreover, $\overline{P}^m{}_j = 0$ gives rise to $\overline{R}^m{}_j = 0$ for vectors on N. Hence, in accordance with (2)

$$dw^m{}_j + w^m{}_\lambda \wedge w^\lambda{}_j = 0. \tag{14}$$

From (12), (13), (14) we see that

$$d\overline{R}_j + \overline{R}_\lambda \wedge w^\lambda{}_j = dR_j + R_h \wedge w^h{}_j.$$

Owing to the definition (6) of a projectively flat 2-space, this completes the proof.

In what follows let X, Y be vectors on N, and Z be the normal. The definition (8) now reads: $D_X Z = X$. Thus,

$$\overline{D}_X Z = X - g(Z, X)Z, \tag{15}$$

where g is seen to be linear in the second slot. Actually, it would be more precise to say that N is umbilical with respect to the normal Z. We now think of N as one of a family of hypersurfaces and postulate that (15) is valid when X is taken equal to Z. In this case $\overline{D}_Z Z$ is proportional to Z and the trajectories of the family whose tangent vector is Z are geodesics. Thus, we have a family of parallel umbilical hypersurfaces [2]. Such hypersurfaces exist when solutions Z of (15) can be found for arbitrary X. The situation is elucidated by the following.

THEOREM. *For $m>2$, equations* (15) *are completely integrable if and only if M is projectively flat and* $g(Z, X)=\bar{w}^\lambda(Z)\overline{R}_\lambda(X)$.

PROOF. In view of (1) the integrability conditions of (15) may be written [3, p. 59]

$$\overline{R}(X, Y)Z = -g(Z, Y)X + g(Z, X)Y - \{Xg(Z, Y) - Yg(Z, X) - g(Z, [X, Y])\}Z. \tag{16}$$

If we first assume that solutions of (15) exist in all directions, the first Bianchi identity may be applied to 3 independent vectors X, Y, Z, m being greater than 2. This allows for a modified form of (16), namely

$$\overline{R}(X, Y)Z = -g(Z, Y)X + g(Z, X)Y - \{g(X, Y) - g(Y, X)\}Z. \tag{17}$$

We notice at once that g must be linear in the first slot also and make $Z=e_\beta$ for simplification. Furthermore, setting $g(e_\beta, X)=g_\beta(X)$, (17) appears in the form

$$\overline{R}^\alpha{}_\beta = -\bar{w}^\alpha \wedge g_\beta - \bar{w}^\lambda \wedge g_\lambda\delta^\alpha{}_\beta.$$

From (3) it is now easily seen that $\overline{R}_\beta=g_\beta$ and (5) gives the desired result, $\overline{P}^\alpha{}_\beta=0$. In addition, $g(Z, X)=\bar{w}^\lambda(Z)\overline{R}_\lambda(X)$.

For the converse we write (15)

$$\bar{w}^\lambda{}_\beta = \bar{w}^\lambda - \overline{R}_\beta\delta^\lambda{}_\beta. \tag{18}$$

Similarly, the integrability conditions (16) become

$$\overline{R}^\alpha{}_\beta = -\bar{w}^\alpha \wedge \overline{R}_\beta - d\overline{R}_\beta\delta^\alpha{}_\beta. \tag{19}$$

But now $\overline{P}^\alpha{}_\beta=0$, implying that

$$\overline{R}^\alpha{}_\beta = -\bar{w}^\alpha \wedge \overline{R}_\beta - \bar{w}^\lambda \wedge \overline{R}_\lambda\delta^\alpha{}_\beta. \tag{20}$$

Because of (18) we find that $\overline{R}_\lambda \wedge \bar{w}^\lambda{}_\beta = -\bar{w}^\lambda \wedge \overline{R}_\lambda$ and then, since (6) holds, (19) is a consequence of (20).

Lastly, from the preceding material the reader may supply the proof of the

THEOREM. *For $m=2$, equations* (18) *are completely integrable if and only if M is projectively flat.*

ADDENDUM. In comparison with spaces of constant curvature [**2**], one might be inclined to multiply the vector e_m occurring in the definition (8) by some factor. This, however, would not materially alter our proofs because of the absence of a metric. Similarly, we could use the additional definition $w^m{}_m=0$ for vectors on N, which means that the derivative of e_m is tangent to N. Then, on account of (1), $w^m{}_h \wedge w^h=0$, and owing to the definition (8) we infer from (2) that $\overline{R}^m{}_m=0$. Since $\overline{P}^m{}_m=0$, (5) and (4) now tell us that $\overline{R}^\lambda{}_\lambda=0$, and because of (9), we also find $R^h{}_h=0$. This means that the connection of N preserves volume [**5**, p. 144].

ACKNOWLEDGMENT. The author is indebted to the referee for the addendum.

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