SYMMETRIC REPRESENTATIONS OF NONDEGENERATE GENERALIZED n-GONS

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1. **Introduction.** Let P_n be a nondegenerate generalized polygon with $s+1$ points on each line and $s+1$ lines through each point (cf. [1] for definitions), and suppose $s>1$. Then by Theorem 1 of [1], $n=3$, 4, or 6, and examples for these parameters are known for any prime power s. If $n=3$, P_n is a projective plane, and a desarguian plane always has a symmetric incidence matrix. If $n=4$ and $s=2$ the essentially unique P_4 has a symmetric incidence matrix [4]. We ask: When may P_4 have a symmetric incidence matrix A, and in that case what can we say about A? The principal results of this paper[1] are:

THEOREM 1. *If $n=6$, P_n has no symmetric incidence matrix.*

THEOREM 2. *If A is a symmetric incidence matrix of a P_4, then the minimal polynomial for A is $f(x)=(x-(s+1))(x^2-2s)x$. Let r_i be the multiplicity of θ_i as a root of the characteristic polynomial for A, $\theta_1=s+1$, $\theta_2=(2s)^{1/2}$, $\theta_3=-(2s)^{1/2}$, and $\theta_4=0$. Then $r_1=1$, $r_4=\frac{1}{2}s(1+s^2)$, and $r_2+r_3=\frac{1}{2}s(1+s)^2$. Also, $\operatorname{tr}(A)=1+s+(2s)^{1/2}(r_2-r_3)$, so that if $(2s)^{1/2}$ is irrational, $r_2=r_3=\frac{1}{4}s(1+s)^2$. (In view of [6], s must be a prime power, so that $(2s)^{1/2}$ will be irrational except when s is an odd power of 2.)*[2]

It remains open whether or not P_4 always has a symmetric incidence matrix, however, there is always a normal one.[2] In the case $s=2$, the particular symmetric incidence matrix considered has $\operatorname{tr}(A)=1+s^2$, and characteristic polynomial $F(x)=(x-3)(x-2)^5 \cdot(x+2)^4x^5$. Perhaps of independent interest is

THEOREM 3. *If M is a set of points of P_4 (embedded in $PG(3, s)$ as in [6]) no two of which are collinear, then $|M|\leqq 1+s^2$, and there is an M with $|M|=1+s^2$.*

THEOREM 4. *For the case $n=3$, the incidence matrix A may be assumed to be symmetric at least if P_n is desarguian, and then has minimal polynomial $f(x)=(x-(s+1))(x^2-s)$. Let r_i be the multiplicity of θ_i as a root of the characteristic polynomial of A, $\theta_1=s+1$, $\theta_2=\sqrt{s}$, $\theta_3=-\sqrt{s}$.*

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[2] See Addendum.

Then $\operatorname{tr}(A)=1+s+(r_2-r_3)\sqrt{s}$. *Also* $r_1=1$, *and if* $\sqrt{s}$ *is irrational*, $r_2=r_3=\frac{1}{2}s(1+s)$.

2. **The case** $n=6$. Assume A is a symmetric incidence matrix of P_6. From Lemmas 3.4 and 6.1 of [1] it follows that the characteristic polynomial $F(x)$ of $A^2=A^TA$ is

(1) $F(x)=(x-(s+1)^2)(x-3s)^{k_1}(x-s)^{k_2}x^{k_3}$, where $k_1=s(1+s^2)\cdot(1+s+s^2)/6$, $k_2=\frac{1}{2}s(1+s)^2(1-s+s^2)$, $k_3=s(s^4+s^2+1)/3$. From Lemma 6.1 of [1] it also follows that

(2) $A^6=4sA^4-3s^2A^2+(s+1)J$, where J is the matrix of order v with all entries equal to 1, $v=1+s+s^2+s^3$.

Let $\lambda_1, \cdots, \lambda_v$ be the characteristic values of A, $|\lambda_1|\geqq|\lambda_2|\geqq\cdots$. By the Weyl inequalities [2, p. 116], since $s+1$ is clearly a characteristic value of A, we have $|\lambda_j|\leqq(3s)^{1/2}$ for $j>1$. So $s+1$ is a simple root of the characteristic polynomial of A, and the only root with absolute value equal to $s+1$. Since $(1+s)^{-1}A$ is doubly stochastic, it follows from 5.3.1 [2, p. 123] that A is indecomposable. So by 5.2.7 [2, p. 123], for each pair (i, j), $1\leqq i, j\leqq v$, there is a k less than the degree of the minimal polynomial of A such that the (i, j) entry of A^k is positive.

LEMMA 2.1. *The minimal polynomial for* A *is* $f(x)=(x-(s+1))\cdot(x^2-3s)(x^2-s)x$.

PROOF. Using (1), the fact that $f(x)$ can have no repeated roots, $-(s+1)$ is not a root of $f(x)$, and that the degree of the minimal polynomial $f(x)$ is at least 6, the lemma follows. We have yet only to establish that the degree of $f(x)$ must be at least 6. By the remarks preceding the lemma, we need only find a pair (i, j) such that the (i, j) entry of A^k is zero provided $1\leqq k\leqq 4$. Let L_i be the line indexing row i of A. Then in the notation of [1], we need only find a subscript j such that if L_j, x_j are the line and point indexing row j and column j, respectively, of A, then $\lambda(L_i, L_j)>4$ and $\lambda(L_i, x_j)>3$. But $\lambda(L_i, L)\leqq 4$ for $1+s+\cdots+s^4$ different lines L, and $\lambda(L_i, x)\leqq 3$ for $1+s+s^2+s^3$ different points x. This leaves at least $(1+s+\cdots+s^5)-(2(1+s+s^2+s^3)+s^4)>0$ suitable j's.

From (2) and Lemma 2.1 it follows that

(3) $A^5=4sA^3-3s^2A+J$.

From Lemma 3.2 of [1] it follows that

(4) $\operatorname{tr}(A^4)=(1+2s)(1+s)^2(1+s^2+s^4)$.

Now let $\theta=s+1$, $f_0(x)=f(x)/x-\theta$, so by Lemma 3.4 of [1], $\operatorname{tr}(f_0(A))=f_0(\theta)$. After computing this we find

(5) $\operatorname{tr}(A^5)=4s\operatorname{tr}(A^4)-3s^2\operatorname{tr}(A)+(1+s+\cdots+s^5)$.

Using (3), (4) and (5) we find

(6) $\operatorname{tr}(A^3)=\operatorname{tr}(A^4)=(1+2s)(1+s)(1+s+\cdots+s^5)$. However, it is readily verified that $\operatorname{tr}(A^3)\leqq(s+1)^2(1+s+\cdots+s^5)$, a contradiction. This completes a proof of Theorem 1.

3. **The case** $n=4$. Let A be a symmetric incidence matrix of P_4. By an argument analogous to that used in the case $n=6$, we find that

LEMMA 3.1. *The minimal polynomial for A is $f(x)=(x-(s+1))\cdot(x^2-2s)x$. Let $\theta_1=s+1, \theta_2=(2s)^{1/2}, \theta_3=-(2s)^{1/2}, \theta_4=0$, and let r_i be the multiplicity of θ_i as a root of the characteristic polynomial of A. Then $r_1=1, r_4=\frac{1}{2}s(1+s^2)$, and $r_2+r_3=\frac{1}{2}s(1+s)^2$.*

Put $f_i(x)=f(x)/x-\theta_i$, so that $\operatorname{tr}(f_i(A))=f_i(\theta_i)\cdot r_i$. Using $i=1$ we find

(7) $\operatorname{tr}(A^3)=2s\operatorname{tr}(A)+(s+1)(s^2+1)$.

From $\operatorname{tr}(f_2(A)+f_3(A))=f_2(\theta_2)\cdot r_2+f_3(\theta_3)\cdot r_3$, we have

(8) $\operatorname{tr}(A^3)=2s(2s)^{1/2}(r_2-r_3)+(s+1)^3$.

Then (7) and (8) imply

(9) $\operatorname{tr}(A)=(2s)^{1/2}(r_2-r_3)+1+s$.

Then from (9) it is clear that if $(2s)^{1/2}$ is irrational, $r_2=r_3$, and Theorem 2 is proved.

If A is the natural incidence matrix of the P_4 with $s=2$ listed in [4], then $\operatorname{tr}(A)=5=1+s^2$. So using (9) and Lemma 3.1 we may calculate the characteristic polynomial to be $(x-3)(x-2)^5(x+2)^4x^5$.

Now let A be any incidence matrix of P_4 embedded in $PG(3, s)$. Then $A^TA-sI=B_0$ is an incidence matrix of a projective geometry $G=PG(3, s)$. Furthermore, $B_0=PB$ where the following hold (cf. [3], [6]):

(i) If column j of B is indexed by a point x of G, x a 1-dimensional subspace of 4-tuples over $GF(s)$, then row j of B is indexed by the null space Nx of x.

(ii) PB is symmetric with 1's on the main diagonal, and P represents a collineation of G induced by a 4×4 nonsingular skewsymmetric matrix C (with zeros on the main diagonal).

If D is a nonsingular matrix over $GF(s)$ such that $DCD^T=E$, where E is the direct sum of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with itself, then D yields a collineation R^T of B such that $AR=QB$, where Q is the collineation of B induced by E (cf. Theorem II.3 of [3]). Similar remarks apply to A^T. Using the essential uniqueness of $PG(3, s)$ and $GF(s)$, it follows that there must be a permutation matrix R^* such that $(A^TR^*)^T(A^TR^*)=(AR)^T(AR)$. Then $AR(R^*)^T$ is a normal incidence matrix of P_4.

Let A be any incidence matrix of P_4. Let $\lambda_1, \cdots, \lambda_v$ be the characteristic values of A, so that $\lambda_1=s+1$, $0\neq|\lambda_j|\leqq(2s)^{1/2}$ for $2\leqq j\leqq 1+\frac{1}{2}s(1+s)^2$, and $\lambda_j=0$ otherwise. By Shur's inequality $|\lambda_j|=(2s)^{1/2}$ for $2\leqq j\leqq 1+\frac{1}{2}s(1+s)^2$ if and *only if* A is normal.

Suppose that A is normal. Then A is associated with a collineation of P_4 as follows. Let x_i and L_i be the point and line indexing column i and row i, respectively, of A, $1\leqq i\leqq v$. Let L_j be any line containing, say, points $x_{i_0}, x_{i_1}, \cdots, x_{i_s}$. Then lines $L_{i_0}, \cdots, L_{i_s}$ must meet a point $x_{j'}$. For arbitrary indices i, j, $x_j\in L_i$ if and only if $x_{i'}\in L_j$ if and only if $x_{j'}\in L_{i'}$. Thus $x\rightarrow x_{i'}$ is a collineation of P_4 such that $L_i\rightarrow L_{i'}$. A is symmetric if and only if $i\rightarrow i'$ is the identity permutation on $1, \cdots, v$. We know little about $\operatorname{tr}(A)$, except that $\operatorname{tr}(A)\leqq t$ where t is the maximum possible number of points of P_4 no two of which are collinear.

In connection with this we prove Theorem 3.

PROOF. Since P_4 has $(1+s)(1+s^2)$ lines with $1+s$ lines through each point, clearly $|M|\leqq 1+s^2$. To construct an M with $|M|=1+s^2$, we use an observation derived from the fact that in $PG(3, s)$ the intersection of any two distinct planes is a line, and also that the set of lines through a given point of P_4 form a plane in the $PG(3, s)$ in which P_4 is embedded.

REMARK. Let x, y be any two points of P_4, and let $w_0, \cdots, w_s$ be the points collinear with both x and y. Then any point z collinear with at least two of the w_i's is collinear with all of them.

Now to construct M with $|M|=1+s^2$. Let P be any point of P_4, L a line through P, $z_1, \cdots, z_s$ the other points of L. Let $L_{i1}, \cdots, L_{is}$ be the lines through z_i other than L, $1\leqq i\leqq s$, and let x be any point of L_{11} different from z_1. Then x determines a *layer* $\mathfrak{L}_x$ of points, one on each L_{ij}, $1\leqq i, j\leqq s$, as follows:

$$A = \{z \mid z\in L_{ij},\ 2\leqq i<s,\ 1\leqq j\leqq s, \text{ and } x \text{ and } z \text{ are collinear}\},$$

$$B = \{z \mid z\in L_{1j},\ 1\leqq j\leqq s, \text{ and } z \text{ is collinear with some point in } A\}.$$

Define $\mathfrak{L}_x$ by $\mathfrak{L}_x=A\cup B$. Let z be the point of some L_{ij}, $2\leqq i\leqq s$, $1\leqq j\leqq s$, such that x and z are collinear. Then for any j', $2\leqq j'\leqq s$, let $z'\in L_{1j'}$ be the point such that z and z' are collinear. Since z' is collinear with two of the points (z and z_1) which are collinear with x and z_2, z' must be collinear with all of the points collinear with both x and z_2. By similar arguments with the other L_{ij} playing the role of L_{11} we see that $\mathfrak{L}_x$ is a set of s^2 points, one on each L_{ij}, such that any point of $\mathfrak{L}_x$ on some L_{ij} is collinear with just those points of the $L_{i'j'}$, $i' \neq i$, $1 \leqq j' \leqq s$, which are in $\mathfrak{L}_x$. It follows that each x' in $\mathfrak{L}_x$ completely determines $\mathfrak{L}_x$, and for $x, y \in L_{ij}$, the layers containing x and y are disjoint unless $x = y$. There are s different layers $\mathfrak{L}_1, \cdots, \mathfrak{L}_s$ corresponding to the s points $x_1, \cdots, x_s$ of L_{11} different from z_1. If a point of $\mathfrak{L}_a$ and a point of $\mathfrak{L}_b$ are collinear, they must lie on the same L_{ij}. Thus we obtain a set M of $1+s^2$ pairwise noncollinear points: $M = \bigcup_{1 \leq i,j \leq s} (\mathfrak{L}_i \cap L_{ij}) \cup \{P\}$.

Interpreted for the 4-dimensional vector space over $GF(s)$, Theorem 3 says:

COROLLARY. *Given any nonsingular skewsymmetric* 4×4 *matrix* C (*with zeros on the main diagonal*) *over* $GF(s)$, *there is a set* M *of pairwise independent* 4×1 *column vectors over* $GF(s)$ *with* $|M| = 1+s^2$ *and such that for any* $x, y \in M$, $y^T Cx = 0$ *if and only if* $y = x$.

4. **The case** $n=3$. The usual way of obtaining an incidence matrix A of a projective plane from a 3-dimensional vector space yields a symmetric one: x is in the null space of y if and only if y is in the null space of x. Any A is the incidence matrix of a (v, k, λ)-configuration with $\lambda = 1$, so A is normal [5] and the characteristic polynomial of $A^T A$ is well known to be $(x-(s+1)^2)(x-s)^{s(1+s)}$. The steps leading to the minimal polynomial for symmetric A are analogous to those for $n=6$ and $n=4$. To complete the proof of Theorem 4 requires a step similar to (8), and we leave the details to the reader.

5. **Addendum.** (Added in proof September 20, 1968.) Singleton's proof of the uniqueness of P_4 is in error. Moreover, the examples of Benson [7] in the odd characteristic case may be shown to be inequivalent to those of Singleton (cf. [8]). Also, we have recently observed that if A is symmetric and $n=4$, it necessarily follows that $\operatorname{tr}(A) = 1+s^2$. Thus $2s$ is a perfect square and r_2 and r_3 may be calculated. Consequently for odd characteristic both the examples of Benson and those of Singleton fail to have symmetric incidence matrices.

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