SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A SIMPLE EXISTENCE AND UNIQUENESS PROOF FOR A SINGULAR CAUCHY PROBLEM

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Consider the singular Cauchy problem in $m+1$ space-time variables $(x, t) = (x_1, x_2, \cdots, x_m, t)$ with u_i and u_{ij} denoting first and second space-derivatives,

$$u_{tt} + kt^{-1}u_t = F(x, t, u, u_t, u_i, u_{it}, u_{ij}) \qquad (t > 0), \tag{1}$$

$$u(x, 0) = f(x), \qquad u_t(x, 0) = 0. \tag{2}$$

The functions F and f are analytic in their arguments.

THEOREM. *The singular Cauchy problem* (1, 2) *has for* $k \geqq 0$ *a unique solution among the class of analytic functions.*

Equation (1) is a generalization of the Euler-Poisson-Darboux (EPD) equation $\Delta u = u_{tt} + kt^{-1}u_t$, whose study by A. Weinstein in 1952 [*On the wave equation and the equation of Euler-Poisson*, Amer. Math. Soc. Proc. 5th Sympos. Appl. Math., McGraw-Hill, New York, 1954, pp. 137–147] has given rise to literature too extensive to be listed here.

The proof makes use of the classic Cauchy-Kowalewski Theorem whose proof, when suitably modified, serves to establish this theorem also. Consider the formal power series

$$u(x, t) = \sum c_{\alpha\lambda} x^\alpha t^\lambda \qquad (\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)), \tag{3}$$

where x^α indicates the product of the x_i's and their corresponding powers α_i; summation is over nonnegative powers. The initial data (2) and successive space-differentiations yield all coefficients of the form $c_{\alpha 0}$ and $c_{\alpha 1}$. Now write

$$u_{tt} + kt^{-1}u_t = \sum \lambda(\lambda + k - 1) c_{\alpha\lambda} x^\alpha t^{\lambda-2} = F(x, t, u, \cdots), \tag{4}$$

and from this is determined $c_{\alpha 2}$. Every $c_{\alpha\lambda}$ can be expressed as a poly-

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nomial P with positive coefficients in the $c_{\beta\mu}$ and the partial derivatives of F (not denoting the dependency of P on α, λ)

$$(\lambda + k - 1)c_{\alpha\lambda} = (\lambda - 1)P_{\alpha\lambda}(c_{\beta\mu}; F) \qquad (\lambda > 1, \mu < \lambda). \tag{5}$$

If $k=0$, then (1, 2) becomes a regular Cauchy problem, and (3) is, by the Cauchy-Kowalewski Theorem, convergent. Moreover, it is clear from (4) that the coefficients in the series solution of the regular equation are given by (5), with $k=0$. An induction on λ proves that $|c_{\alpha\lambda}|$ is no greater than the absolute value of the corresponding coefficient in the case $k=0$. Hence the series (3) is absolutely and uniformly convergent for $k \geqq 0$, and represents the solution of singular problem (1, 2).

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