# TWO THEOREMS OF EULER AND A GENERAL PARTITION THEOREM

GEORGE E. ANDREWS[1]

**1. Introduction.** Euler proved that the partitions of a natural number $n$ into distinct parts are equinumerous with the partitions of $n$ into odd parts [2, p. 277]. A second theorem due to Euler states that every natural number is uniquely representable as a sum of distinct powers of 2 [2, p. 277].

If $S_1$ and $S_2$ are subsets of the natural numbers $N$, let us call $(S_1, S_2)$ an Euler-pair if for all natural numbers, $n$, the number of partitions of $n$ into distinct parts taken from $S_1$ equals the number of partitions of $n$ into parts taken from $S_2$. Euler's first theorem may then be stated by saying $N$ and $\{n \in N \mid 2 \nmid n\}$ are an Euler-pair, and his second theorem may be stated by saying that $\{2^n \mid n \in N \text{ or } n=0\}$ and $\{1\}$ are an Euler-pair. Other examples of Euler-pairs are

$$(\{n \in N \mid 3 \nmid n\}, \{n \in N \mid n \equiv 1, 5 \pmod 6\})$$

due to I. J. Schur [3, p. 495], and

$$(\{n \in N \mid n \equiv 2, 4, 5 \pmod 6\}, \{n \in N \mid n \equiv 2, 5, 11 \pmod{12}\})$$

due to H. Göllnitz [1, p. 175]. The object of this paper is to give a simple characterization of Euler-pairs.

Throughout this paper all sets $S_i$ which we consider will be subsets of the natural numbers $N$. The notation $mS_i$ denotes the set $\{mn \mid n \in S_i\}$. The notation $S_i - S_j$ denotes the set $\{n \in S_i \mid n \notin S_j\}$. By $p(S_i; n)$ we denote the number of partitions of $n$ into parts taken from $S_i$. By $q(S_i; n)$ we denote the number of partitions of $n$ into distinct parts taken from $S_i$. We shall write $S_i = \{s_1(i), s_2(i), s_3(i), \cdots\}$ where the elements are arranged in ascending order of magnitude.

THEOREM 1. *$(S_1, S_2)$ is an Euler pair if and only if $2S_1 \subseteq S_1$ and $S_2 = S_1 - 2S_1$.*

In §2 we shall prove Theorem 1. In §3 we examine some of the corollaries of Theorem 1.

**2. Proof of Theorem 1.** First we require the following result.

LEMMA. *If for every natural number $n$, $p(S_1; n) = p(S_2; n)$, then*

Received by the editors November 14, 1967.

[1] Partially supported by National Science Foundation Grant GP-6663.

$S_1=S_2$. *Indeed if $m$ is the least integer in* $(S_1\cup S_2)-(S_1\cap S_2)$, *then* $p(S_1;m)\neq p(S_2;m)$.

PROOF. We need only show that the second statement is valid. Without loss of generality we assume $m=s_r(1)$. Now $p(S_1;m)$ is just the number of partitions of $m$ into parts taken from the set $\{s_1(1), s_2(1), \cdots, s_r(1)\}$. On the other hand, $p(S_2;m)$ is just the number of partitions of $m$ into parts taken from $S_2$ which do not exceed $m$ in magnitude; consequently by the definition of $m$, $p(S_2;m)$ is just the number of partitions of $m$ into parts taken from the set $\{s_1(1), s_2(1), \cdots, s_{r-1}(1)\}$. Therefore $p(S_1;m)=p(S_2;m)+1$. This completes the proof of the lemma.

PROOF OF THEOREM 1. First we treat sufficiency. The generating function for $q(S_1;n)$ is $\prod_{n\in S_1}(1+q^n)$, which is absolutely and uniformly convergent for $|q|\leqq 1-\delta$.

The generating function for $p(S_2;n)$ is $\prod_{n\in S_2}(1-q^n)^{-1}$, again absolutely and uniformly convergent for $|q|\leqq 1-\delta$.

Now assuming $2S_1\subseteq S_1$ and $S_2=S_1-2S_1$, we have

$$\prod_{n\in S_1}(1+q^n)=\prod_{n\in S_1}(1-q^{2n})(1-q^n)^{-1}$$
$$=\prod_{n\in S_1-2S_1}(1-q^n)^{-1}=\prod_{n\in S_2}(1-q^n)^{-1}.$$

This establishes that $q(S_1;n)=p(S_2;n)$ for all $n$. Therefore $(S_1, S_2)$ is an Euler-pair.

Next we treat necessity. We suppose that $(S_1, S_2)$ is an Euler-pair. If $(S_1, S_3)$ were also an Euler-pair, then by the lemma and the definition of Euler-pair we have $S_2=S_3$. If we can show $2S_1\subseteq S_1$, then as above we know that $(S_1, S_1-2S_1)$ is an Euler-pair and hence $S_2=S_1-2S_1$. Thus we need only show that $2S_1\subseteq S_1$.

Suppose $2S_1\not\subseteq S_1$. Let $s_r(1)$ be the least element of $S_1$ such that $2s_r(1)\notin S_1$. Now

$$\prod_{n\in S_2;n<2s_r(1)}(1-q^n)^{-1}\prod_{n\in S_2;n\geq 2s_r(1)}(1-q^n)^{-1}$$
$$=\prod_{n\in S_1}(1+q^n)\quad=\prod_{n\in S_1}(1-q^{2n})(1-q^n)^{-1}$$
$$=\prod_{n\in S_1;n<s_r(1)}(1-q^{2n})\prod_{n\in S_1;n<2s_r(1)}(1-q^n)^{-1}\prod_{n\in S_1;n\geq s_r(1)}(1-q^{2n})$$
$$\cdot\prod_{n\in S_1;n>2s_r(1)}(1-q^n)^{-1}=\prod_{n\in S_1-2S_1;n<2s_r(1)}(1-q^n)^{-1}$$
$$\cdot\prod_{n\in S_1;n\geq s_r(1)}(1-q^{2n})\prod_{n\in S_1;n>2s_r(1)}(1-q^n)^{-1}.$$

Thus from the above identity we see that

$$\prod_{n\in S_2;\, n<2s_r(1)} (1-q^n)^{-1} \quad\text{and}\quad \prod_{n\in S_1-2S_1;\, n<2s_r(1)} (1-q^n)^{-1}$$

agree as power series in $q$ for the first $2s_r(1)$ coefficients. However, if these two functions were unequal, then by the lemma they would have unequal coefficients among the first $2s_r(1)$ coefficients; hence these products are equal. Thus, cancelling them in the above equation, we obtain

$$\prod_{n\in S_2;\, n\geq 2s_r(1)} (1-q^n)^{-1} = \prod_{n\in S_1;\, n\geq s_r(1)} (1-q^{2n}) \prod_{n\in S_2;\, n>2s_r(1)} (1-q^n)^{-1}.$$

Consider now the coefficient of $q^{2s_r(1)}$ in the power series expansion of both sides of this equation. On the left-hand side it is either 0 or 1, and on the right-hand side it is $-1$. Thus we have a contradiction, and therefore we must have $2S_1 \subseteq S_1$. This completes the proof of Theorem 1.

**3. Corollaries.** We start with the following special case of Theorem 1.

THEOREM 2. *Let $S_1 \subseteq N$ be such that $n\in S_1$ if and only if $2n \in S_1$. Let* $S_2 = \{n\in S_1 \mid n\equiv 1 \pmod 2\}$. *Then $(S_1, S_2)$ is an Euler-pair.*

PROOF. We need only show

$$S_1 - 2S_1 = \{n \in S_1 \mid n \equiv 1 \pmod 2\}.$$

Clearly the set on the left-hand side contains the set on the right-hand side. On the other hand, if $n\in S_1$ and $n=2m$, then by hypothesis $m\in S_1$, hence $n\in 2S_1$; consequently $n\notin S_1-2S_1$. Thus we have Theorem 2.

Both of Euler's theorems are obvious consequences of Theorem 2, as is Schur's result. Göllnitz's theorem is not a corollary of Theorem 2, but it is easily deduced from Theorem 1. We may also prove many other partition theorems which do not seem to have been noted.

THEOREM 3. *The number of partitions of a natural number $n$ into distinct parts, each of which is representable as the sum of two squares, equals the number of partitions of $n$ into odd parts each of which is representable as the sum of two squares.*

PROOF. By [2, p. 299], $n$ is representable as the sum of two squares if and only if $2b$ is. The desired result now follows from Theorem 2.

THEOREM 4. *If $p$ is a prime* $\equiv 1, 7 \pmod 8$, *then the number of partitions of a number $n$ into distinct quadratic residues* (mod $p$) *equals the* *number of partitions of $n$ into odd parts which are quadratic residues* (mod $p$).

PROOF. Since $p \equiv 1$, 7 (mod 8), 2 is a quadratic residue (mod $p$) [2, p. 75]. Thus $n$ is a quadratic residue (mod $p$) if and only if $2n$ is a quadratic residue (mod $p$). The desired result now follows from Theorem 2.

## REFERENCES

1. H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225** (1967), 154–190.

2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

3. I. J. Schur, *Zur additiven Zahlentheorie*, S.-B. Akad. Wiss. Berlin (1926), 488–495.

PENNSYLVANIA STATE UNIVERSITY