DERIVATIONS OF THE LIE ALGEBRA OF POLYNOMIALS UNDER POISSON BRACKET

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Abstract. We exhibit a class of outer derivations of the Lie algebra P of complex polynomials under Poisson bracket, and prove that every derivation of P is a linear combination of one of these and an inner derivation, although this decomposition may not be unique. In particular, we show that any derivation of P which maps constants to zero must be inner. We use these results to characterise certain solutions of the Dirac problem.

1. Introduction. Let E denote the collection of all C^∞ complex-valued functions of $2n$ real variables $x=(q_1, \cdots, q_n, p_1, \cdots, p_n)$, and define the *Poisson bracket* of two elements of E to be

$$\{f, g\} = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial q_j}\cdot\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\cdot\frac{\partial g}{\partial q_j}\right).$$

It is well known [1] that this defines a Lie bracket on E, and we shall denote the corresponding Lie algebra also by E. The collection of all complex polynomials in the variables x forms a Lie subalgebra of E, which we shall denote by P. A *derivation of* P is a linear map $D: P \to P$ such that

$$(1) \qquad \{D(f), g\} + \{f, D(g)\} = D(\{f, g\}) \qquad \text{all } f, g \in P.$$

A *derivation of* P *into* E is a linear map $D: P \to E$ such that (1) holds. P possesses *inner* derivations of form [2]

$$D(f) = \{A, f\} \qquad \text{some } A \in P \quad \text{and all} \quad f \in P.$$

A derivation of P which is not of this form will be called *outer*.

In this paper we find all the derivations of P. We first exhibit (§2) a class of outer derivations of P, and in §3 we prove that every derivation of P is a linear combination of one of these and an inner derivation, although this decomposition may not be unique. In particular, we show that a derivation of P which maps constants to zero must be inner. Analogous results hold for derivations of P into E. Finally in §4 we characterise certain solutions of the Dirac problem.

2. A class of outer derivations of P. It is not obvious at the outset that P does in fact possess outer derivations, but we have

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LEMMA 1. *For any set of complex numbers* $a=(a_1, \cdots, a_n)$, *the map*

$$D_a: f \to f - \sum_{j=1}^{n}\left(a_j p_j \frac{\partial f}{\partial p_j} + (1-a_j) q_j \frac{\partial f}{\partial q_j}\right) \tag{2}$$

is an outer derivation of P.

PROOF. Condition (1) can be verified directly, although this is tedious. A more interesting proof is as follows. For each $f\in P$, let X_f be the vector field on R^{2n} defined by

$$X_f = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}\right).$$

Then for all $f, g\in P$ we have [3]

$$[X_f, X_g] = X_f X_g - X_g X_f = -X_{\{f,g\}}, \qquad X_f g = \{g, f\} = -\{f, g\}.$$

Let Ω denote the differential 2-form $\sum_{j=1}^{n} dp_j\wedge dq_j$, defined by

$$\Omega(X, Y) = \sum_{j=1}^{n}((Xp_j)(Yq_j) - (Yp_j)(Xq_j))$$

for all vector fields X, Y. Let ω be any differential 1-form such that $d\omega=\Omega$. This property is expressed by the equation [3]

$$\Omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \qquad \text{all } X, Y.$$

Now let $X=X_f$, $Y=X_g$. After some simple rearranging we get

$$\{f, g\} - \omega(X_{\{f,g\}}) = \{f - \omega(X_f), g\} + \{f, g - \omega(X_g)\} \qquad \text{all } f, g$$

showing that the map $f\to f-\omega(X_f)$ is a derivation of P. We obtain D_a by putting $\omega=\sum_{j=1}^{n}(a_j p_j dq_j-(1-a_j)q_j dp_j)$. Also, since $D_a(1)\neq 0$, we see that D_a is not inner.

LEMMA 2. *The smallest Lie subalgebra of* P *containing the collection of polynomials*[1] $\pi=\{q_i, q_i^2, q_iq_j, q_i^3, p_i^2\}$ *is* P *itself*.

PROOF. Let Q be the smallest such subalgebra. It is not hard to see that, for each i, Q contains all the functions of q_i and p_i only. From the equations

$$\{q_rq_{r+1}, q_rp_r^2\} = 2q_rq_{r+1}p_r, \qquad \{q_1q_2\cdots q_r, q_rq_{r+1}p_r\} = q_1q_2\cdots q_rq_{r+1}$$

[1] Here and in the sequel i and j take the values $1, 2, \cdots, n$.

and a simple induction argument, we find that $q_1q_2 \cdots q_n \in Q$. Each $f \in P$ of form $f_1f_2 \cdots f_n$, where each f_i is a function of q_i and p_i only, can be expressed as

$$f = \{ \cdots \{\{q_1q_2 \cdots q_n, g_1\}, g_2\} \cdots, g_n\}$$

where each g_i is a function of q_i and p_i only, such that $\partial g_i/\partial p_i = f_i$. Thus Q contains all polynomials of product form. Taking linear combinations, we obtain $Q = P$.

3. **All the derivations of** P. The inner derivations of P can be characterised simply by

Lemma 3. *If D is a derivation of P, and $D(1)=0$, then D is inner.*

Proof. From (1) we have

$$\{D(q_i), p_j\} + \{q_i, D(p_j)\} = D(\{q_i, p_j\}) = 0$$

hence,

$$\frac{\partial}{\partial q_j} D(q_i) = -\frac{\partial}{\partial p_i} D(p_j).$$

Similarly,

$$\frac{\partial}{\partial p_j} D(q_i) = \frac{\partial}{\partial p_i} D(q_j), \qquad \frac{\partial}{\partial q_j} D(p_i) = \frac{\partial}{\partial q_i} D(p_j).$$

Since R^{2n} is simply connected, these conditions imply that there exists a function A such that

$$D(q_i) = -\frac{\partial A}{\partial p_i}, \qquad D(p_j) = \frac{\partial A}{\partial q_j}$$

and $A \in P$ since the $D(q_i)$, $D(p_j)$ are all polynomials. The map

$$S: f \to D(f) - \{A, f\} \qquad \text{all } f \in P$$

is also a derivation of P and has the property

$$S(1) = S(q_i) = S(p_j) = 0; \tag{3}$$

hence, from (1)

$$S\left(\frac{\partial g}{\partial p_i}\right) = S(\{q_i, g\}) = \{q_i, S(g)\} = \frac{\partial}{\partial p_i} S(g) \qquad \text{all } g \in P. \tag{4}$$

Similarly,

$$S\left(\frac{\partial g}{\partial q_j}\right) = \frac{\partial}{\partial q_j} S(g) \qquad \text{all } g \in P. \tag{5}$$

It is clear from (3), (4), (5) that $S(q_i^2) = \text{constant}$, hence from (1) and (4) we have

$$\begin{aligned} 2S\left(q_i \frac{\partial g}{\partial p_i}\right) &= S(\{q_i^2, g\}) = \{S(q_i^2), g\} + \{q_i^2, S(g)\} \\ &= 2q_i \frac{\partial}{\partial p_i} S(g) = 2q_i S\left(\frac{\partial g}{\partial p_i}\right) \qquad \text{all } g \in P. \end{aligned} \tag{6}$$

Substituting $g = q_i p_i,\ q_i^2 p_i,\ q_j p_i$, respectively, into (6) gives

$$S(q_i^2) = S(q_i^3) = S(q_i q_j) = 0.$$

Similarly, we can show that $S(p_i^2) = 0$. Now from (1) we see that $S(f) = S(g) = 0$ implies that $S(\{f, g\}) = 0$. Since $S(f+g)$ is then also zero, this tells us that $S(f) = 0$ for all f belonging to some Lie subalgebra of P containing π, that is, for all $f \in P$. The lemma follows immediately from the definition of S.

LEMMA 4. *If D is a derivation of P into E, and $D(1) = 0$, then D has the form $D(f) = \{A, f\}$ for some $A \in E$ and all $f \in P$.*

PROOF. Identical to that of Lemma 3 except that the $D(q_i)$, $D(p_j)$ are not now necessarily polynomials.

THEOREM 1. *Every derivation D of P is a linear combination of a derivation of type* (2) *and an inner derivation. That is, we can find a polynomial A and complex numbers c, a_i such that $D(f) = cD_a(f) + \{A, f\}$ all $f \in P$.*

REMARK. This decomposition is not necessarily unique, for if D_a and D_b are any two derivations of type (2), we have

$$D_a(f) - D_b(f) = \left\{ \sum_{j=1}^{n} (b_j - a_j) q_j p_j, f \right\} \qquad \text{all } f \in P$$

so in fact, we can replace the D_a in the decomposition by any other D_b merely by redefining A. In case D is inner, the decomposition is unique modulo additive constants in A.

PROOF. From (1) we find that $D(1) = \text{constant} = c$, say. The map

$$T: f \to D(f) - c\left(f - \sum_{j=1}^{n} p_j \frac{\partial f}{\partial p_j}\right) \quad \text{all } f \in P$$

is also a derivation of P, and $T(1)=0$. Hence, by Lemma 3, T is inner. Thus for some constant c and some $A \in P$,

$$D(f) = c\left(f - \sum_{j=1}^{n} p_j \frac{\partial f}{\partial p_j}\right) + \{A, f\} \qquad \text{all } f \in P. \tag{7}$$

THEOREM 2. *Every derivation D of P into E has the form* (7) *for some $A \in E$ and some constant c.*

PROOF. Exactly as for Theorem 1, but using Lemma 4 instead.

These results can be extended to the case of derivations of E (defined in an obvious manner) provided we topologise E suitably so that P forms a dense subset, and make certain continuity assumptions about the derivations. It is obvious from the proofs that Theorems 1 and 2 can be extended to the Lie algebra of real polynomials under Poisson bracket.

4. **Some solutions of the Dirac problem.** A *Dirac map* will mean a solution of the Dirac problem, [5] [6], that is, a linear map L from P into the operators acting on some Hilbert space H, such that $L(1)$ is the identity operator and

$$[L(f), L(g)] = iL(\{f, g\}) \qquad \text{all } f, g \in P.$$

Take $H = L^2(R^{2n})$, and let K denote the C^∞ complex valued functions on R^{2n} with compact support. The condition that the map $f \to L(f)$ defined by

$$L(f)\phi = i\{f, \phi\} + M(f)\phi \qquad \text{all } \phi \in K \tag{8}$$

be a Dirac map, where $M(f)$ is some C^∞ function, is

$$\{M(f), g\} + \{f, M(g)\} = M(\{f, g\}) \qquad \text{all } f, g \in P$$

and

$$M(1) = 1. \tag{9}$$

Thus $f \to M(f)$ must be a derivation of P into E. Using Theorem 2 and condition (9) we obtain

THEOREM 3. *Every Dirac map of type* (8) *is of the form*

$$L(f)\phi = i\{f, \phi\} + \left(f - \sum_{j=1}^{n} p_j \frac{\partial f}{\partial p_j} + \{A, f\}\right)\phi \tag{10}$$

for some $A \in E$.

Souriau's solution [5] corresponds to $A=0$, and was first found by van Hove [7]. Streater [6] obtains the solution corresponding to writing $A = B - \frac{1}{2}\sum_{j=1}^{n} p_j q_j$ in (10). There still remains the question of finding the general Dirac map. Certain other types [4] also reduce to the form (10).

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