SEPARATE CONTINUITY AND MEASURABILITY

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Let X, Y be two locally compact spaces and f a bounded separately continuous complex valued function on $X \times Y$. Glicksberg [2, Theorems 1.2 and 3.1] has shown that if μ, ν are bounded Radon measures on X, Y, then

$$\int\int f(x, y) d\mu(x) d\nu(y) = \int\int f(x, y) d\nu(y) d\mu(x),$$

the inner integral in each case being a continuous function of the remaining variable. In this paper we show that f is a measurable function so that the above equality is actually a particular form of Fubini's theorem (although our proof uses Glicksberg's result). In the second section of the paper, we consider separately continuous maps $f: X \times Y \to Z$, where X, Y are compact, Z is locally compact and there is a measure μ on X with support X. The basic result here is that $f^{-1}(G)$ is a Borel set in $X \times Y$ for each Baire set G in Z.

The advantage of knowing that f is measurable is that it permits considerably greater freedom in handling the repeated integrals. Results of this type have previously been obtained for metrizable spaces. Throughout the paper $\mathbf{C}(X)$ will denote the Banach space of bounded continuous complex valued functions on X and $M(X)$, the space of bounded Radon measures on X.

1. Complex valued functions. We shall denote the support of a measure μ by $\operatorname{supp} \mu$ and, if f is separately continuous on $X \times Y$, $x \in X$, F_x will denote the function $F_x(y) = f(x, y)$ on Y.

PROPOSITION 1.1. *If X, Y are compact, f is a bounded separately continuous complex valued function on $X \times Y$ and $\mu \in M(X)$ then $\{F_x : x \in \operatorname{supp} \mu\}$ is separable in the norm topology of $\mathbf{C}(Y)$.*

PROOF. The proof is an adaptation of part of [3, pp. 131–132]. Replacing X by $\operatorname{supp} \mu$ if necessary, we can assume $X = \operatorname{supp} \mu$. As in §2 of [2], $F_X = \{F_x; x \in X\}$ is weakly compact in $\mathbf{C}(Y)$. Thus by the Kreĭn-Šmulian Theorem [1, p. 434] so is its closed absolutely convex cover $\operatorname{co}(F_X)^-$. This closure is the same whether we use the weak or norm topology on $\mathbf{C}(Y)$. As in the proof of Theorem 3.1 of

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[2] the map $U;\lambda \to \int f(x, y)d\lambda(x)$ is continuous $(\boldsymbol{M}(X), w^*) \to (\boldsymbol{C}(Y), w)$. Since the unit ball in $\boldsymbol{M}(X)$ is the w^* closed convex cover of the point masses it contains, we see that, for each $\lambda \in \boldsymbol{M}(X)$ with $\|\lambda\| \leqq 1$, $U(\lambda) \in \mathrm{co}(F_X)^-$. Thus U is weakly compact. Using Theorem IV.8.9 of [1], we see that the set of characteristic functions of open sets in X is conditionally weakly compact in $L_1(\mu)$. Considering $L_1(\mu)$ as a subspace of $\boldsymbol{M}(X)$ and using Theorem VI.8.12 of [1] we see that $\{\int_G F_x d\mu(x);\ G \text{ open in } X\}$ is conditionally compact in the norm topology of $\boldsymbol{C}(Y)$. If E is the norm closed linear span of this set in $\boldsymbol{C}(Y)$, then E is norm separable and weakly closed.

Let $x_0 \in X$ and let $\mathfrak{N}(x_0)$ denote the directed set of open neighbourhoods of x_0. Then $\mu(N)^{-1}\int_N \phi(x)d\mu(x) \to \phi(x_0)$ $(N \in \mathfrak{N}(x_0))$ for each $\phi \in \boldsymbol{C}(X)$. Applying this result with $\phi(x) = \int f(x, y)d\nu(y)$, $\nu \in \boldsymbol{M}(Y)$, and using [2, Theorem 1.2] we see $\mu(N)^{-1}\int_N F_x d\mu(x) \to F_{x_0}$ in $(\boldsymbol{C}(Y), w)$. Since E is weakly closed, $F_{x_0} \in E$.

Corollary 1.2. *For each $\mu \in \boldsymbol{M}(X)$, $x \to F_x$ is a Borel function from* supp μ *into $\boldsymbol{C}(Y)$, with the norm topology and f is a Borel function* supp $\mu \times Y \to \boldsymbol{C}$.

A function p from one topological space S into another, T, is a *Borel function* if $p^{-1}(B)$ is a Borel set in S whenever B is a Borel set in T. p is a Borel function if and only if $p^{-1}(B)$ is Borel for all open B in T.

Proof. The proof of the first part follows Theorem III.6.11 of [1] with only slight changes. Let $\{\Phi_n\}$ be a countable norm-dense subset of $\{F_x; x \in \operatorname{supp} \mu\}$ and, using the weak* compactness of the unit ball in $\boldsymbol{M}(Y)$, find $\nu_n \in \boldsymbol{M}(Y)$ with $\|\nu_n\| = 1$ and $\int \Phi_n d\nu_n = \|\Phi_n\|$. Then $\|F(x)\| = \operatorname{Sup} |\int F_x d\nu_n|$ is a Borel function on supp μ. By the same argument, $x \to \|F_x - \Phi_n\|$ is Borel measurable for all n and so the inverse image under $x \to F_x$ of any open ball centered on Φ_n in $\boldsymbol{C}(Y)$ is a Borel subset of x. If G is open in $\boldsymbol{C}(Y)$, its inverse image is a countable union of the inverse images of such balls and so is also a Borel set.

Since f can be considered as a composition of functions supp $\mu \times Y \to E \times Y \to \boldsymbol{C}$ given by $(x, y) \to (F_x, y) \to F_x(y)$, the second of which is continuous we need only show that the first is a Borel function.

As E is a separable metrizable space, it has a countable basis $\{G_n\}$ of open sets, so that, if U is open in $E \times Y$, then putting

$$H_n = \{y \in Y;\ G_n \times V \subset U \text{ for some neighbourhood } V \text{ of } y\},$$

we have $U = \bigcup G_n \times H_n$. Hence the inverse image of U under $(x, y) \to (F_x, y)$ is $\bigcup B_n \times H_n$ where B_n is the inverse image of G_n under $x \to F_x$, and so is a Borel set in supp $\mu \times Y$.

EXAMPLE 1.3. In Proposition 1.1 the set $\{F_x; x\in X\}$ need not be norm separable; for example, if $X=Y$ is the one point compactification of an uncountable discrete space and f is the characteristic function of the diagonal with (∞, ∞) removed. Clearly f is separately continuous and yet $\{F_x; x\in X\}$ consists of uncountably many points at a distance one apart.

THEOREM 1.4. *Let X, Y be locally compact; let f be a bounded separately continuous complex valued function on $X\times Y$, and let $\mu\in M(X\times Y)$. Then f is μ measurable.*

PROOF. We can suppose $\mu\geqq 0$. There exists a sequence $\{X_n\times Y_n\}$ of compact sets in $X\times Y$ such that $\mu(X\times Y)=\mu(\bigcup X_n\times Y_n)$. Then for each n, $\mu_n((S)=\mu((X_n\cap S)\times Y)$ defines a bounded measure on the Borel subsets S of X with support in X_n. By 1.2, f is Borel and hence μ-measurable on $\operatorname{supp}\mu_n\times Y_n$ for each n. The result follows since $\mu(X\times Y)=\mu(\bigcup(\operatorname{supp}\mu_n\times Y_n))$.

2. General functions.

PROPOSITION 2.1. *Let X, Y be compact spaces, $\mu\in M(X)$ and suppose $X=\operatorname{supp}\mu$. Let Z be a locally compact space and f a separately continuous function $X\times Y\to Z$. Then $f^{-1}(B)$ is a Borel set in $X\times Y$ for all Baire sets B in Z.*

PROOF. For each bounded continuous complex valued function g on Z, $g\circ f$ is a Borel function by Corollary 1.2. If C is a compact $\mathcal{G}_\delta$ in Z, $C=\bigcap G_i$, say, where the G_i are open in Z, then we can find, for each i, a bounded continuous complex valued function g_n on Z with $g_n(x)=0$ if $x\notin G_n$ and $g_n(x)=1$ if $x\in C$. Since $C=\bigcap g_n^{-1}(1)$ we have $f^{-1}(C)=\bigcup f^{-1}g_n^{-1}(1)=\bigcap(g_n\circ f)^{-1}(1)$, a Borel set.

Using this result we deduce, as in Theorem 1.4.

THEOREM 2.2. *Let X, Y, Z be locally compact spaces, $\mu\in M(X\times Y)$ and f a separately continuous map $X\times Y\to Z$. Then $f^{-1}(B)$ is μ measurable for all Baire sets B in Z.*

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