A COMPLETE AND COUNTABLE ORTHOMODULAR LATTICE IS ATOMIC

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To prove the assertion in the title, let $(L; \leq, ')$ be a countable orthomodular lattice. If L is not atomic, there exists a nonzero $b \in L$ that does not majorize an atom. In L , let $f < g \leq b$. By the orthomodular identity $g = f \vee (f' \wedge g)$ and necessarily $(f' \wedge g) \neq 0$. Since $(f' \wedge g) \leq g \leq b$, $(f' \wedge g)$ can not be an atom and there exists in L a nonzero $d < (f' \wedge g)$. Again $(f' \wedge g) = d \vee (d' \wedge f' \wedge g)$ and $(d' \wedge f' \wedge g) \neq 0$. Let $h = f \vee d$. Since $g = f \vee d \vee (d' \wedge f' \wedge g)$ is an orthogonal join of nonzero elements, it follows that $f < h < g \leq b$. Thus the interval $[0, b] = \{g \mid 0 \leq g \leq b\}$ is order dense-in-itself (i.e., if $f < g$ in $[0, b]$ there exists an $h \in [0, b]$ such that $f < h < g$).

Let C , by Zorn's Lemma, be a maximal chain in $[0, b]$. It follows that C is bounded, countable and order dense-in-itself. Hence C must be order isomorphic to the chain of all rational numbers between any pair of fixed distinct endpoints (see [1, p. 200]). It is well known that such a chain is not order complete. Therefore there exists a $C' \subseteq C$ for which no $\vee C'$ exists in C . If such an element $\vee C'$ exists in L , it must be in $[0, b]$, for b is an upper bound of C' . In that event, by the maximality of C , $\vee C'$ must also be in C . Thus no $\vee C'$ exists in L and L is not complete. Q.E.D.

REFERENCE

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