THE GENERAL PRODUCT OF TWO FINITELY GENERATED ABELIAN GROUPS

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Ito [2] has proved that if a group G is the general product AB where A and B are finite abelian groups then G has a normal subgroup contained either in A or in B and a normal subgroup containing A or B . We use an idea of P. M. Cohn [1] to extend Ito's Theorem to the case where A and B are finitely generated. As a consequence we see that the general product of two finitely generated abelian groups is polycyclic and we incidentally also obtain an easier proof of Cohn's results. Our main theorem is as follows.

THEOREM. *Let $G = AB$ with A and B finitely generated abelian subgroups. Then*

- (I) *G contains a nontrivial normal subgroup contained either in A or in B and*
- (II) *G contains a proper normal subgroup containing A or B .*

PROOF OF (I). Let A^* denote $A \cap G'$ (G' is the commutator subgroup $[G, G]$) and let B^* denote $B \cap G'$. If $A^* > 1$ or $B^* > 1$, then (I) is true, as Ito has already pointed out. Accordingly we assume that $A^* = B^* = 1$ and also that G is infinite in view of Ito's Theorem. Let A_0 be the set of elements $a \in A$ such that there is an element $b \in B$ with $ab \in G'$, and let B_0 be the set of elements $b \in B$ with an $a \in A$ so that $ab \in G'$. We shall show that A_0 and B_0 are isomorphic subgroups. Indeed, since $A^* = B^* = 1$ it is clear that if ab and $ab' \in G'$ with $a \in A_0$ and $b, b' \in B_0$, then $b^{-1}a^{-1}ab' = b^{-1}b' \in B^* = 1$ so that $b = b'$. Thus there is a well-defined map ϕ from A_0 onto B_0 so that if $a \in A_0$, then $a\phi \in B_0$ and $a(a\phi) \in G'$, and for later reference it will be convenient to define the map ψ from A_0 to G' as follows: $a\psi = aa\phi$; it is easy to check that ϕ is one-one and ψ is single valued. Then if $ab, a'b' \in G'$ with $a, a' \in A_0$ and $b, b' \in B_0$, it follows that $ab(b^{-1}a'b'b) = aa'bb' \in G'$; thus A_0 and B_0 are semigroups and ϕ is a semigroup isomorphism. Finally $(babb^{-1})^{-1} = a^{-1}b^{-1}$, and hence A_0 and B_0 are isomorphic groups as we wished to show and G' is the set of elements $a(a\phi)$ with $a \in A_0, a\phi \in B_0$.

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Now we shall show that if for $i = 1, 2$, M_i are normal subgroups of G with $M_2 \leq M_1 \leq G'$, then $M_i\psi^{-1} = A_i$ are subgroups and the index $|M_1:M_2|$ is equal to the index $|A_1:A_2|$. Indeed the fact that A_i is a subgroup follows from the argument above with M_i in place of G' . Then for $a, a' \in A_1$ the cosets $aA_2, a'A_2$ are equal if and only if the cosets $a(a\phi)M_2 = a'(a'\phi)M_2$. For $aA_2 = a'A_2$ if and only if $a^{-1}a' \in A_2$ which holds if and only if $(a\phi)^{-1}a^{-1}a'(a'\phi)$ or $a^{-1}a'(a'\phi)(a\phi)^{-1} \in M_2$ (since $M_2 \triangleleft G$); and this holds if and only if $a'(a'\phi)M_2 = a(a\phi)M_2$. Hence $|M_1:M_2| = |A_1:A_2|$ as was to be shown.

It follows in particular that the order of a finite subgroup of G' normal in G is bounded by the order t of the torsion subgroup of A_0 (or of B_0). Furthermore, since factoring out a normal subgroup of G contained in G' does not alter the hypotheses $A^* = 1 = B^*$, it follows that if M/M_2 is finite with $M_2 \leq M_1 \leq G'$ as before, then the order of M_1/M_2 is bounded by t . Consequently there is a minimal normal subgroup M of G contained in G' ; for in any decreasing sequence of normal subgroups of G , $G' = M_0 > M_1 > \dots > M_i > \dots > M_j > \dots$, there can be at most as many infinite factors M_i/M_{i+1} as the torsion-free rank of A_0 and the finite indices $|M_i:M_j|$ are bounded by t .

Let A_M denote $M\psi^{-1}$ and B_M denote $A_M\phi$. Modulo M , $A_M = B_M$ and hence A_M is central in G (since $G = AB$). Thus $A_MB_M \triangleleft G$ and $M = A_MB_M \cap G'$. If A_MB_M is not abelian there is an $a^{-1}b \in M$ with $a \in A_M$ so that $[a, B_M] \neq 1$. Let S be the subgroup generated by a and G' . Then S is normal in G and so also is the commutator subgroup S' . Furthermore $S' \leq G' \cap A_MB_M = M$ since $a \in A_M$ and $A_MB_M \triangleleft G$. Hence by the minimality of M , $S' = M$. Now in general if S is the extension of a normal abelian subgroup C by a cyclic group generated by a , then S' is the set T of commutators $\{[a, c], c \in C\}$. For $[a, c_1c_2] = [a, c_1][a, c_2]$ so that $[a, c^{-1}] = [a, c]^{-1} = [c, a]$ and hence T is a subgroup. Furthermore, from the fact that $L = [a^{-1}a, c] = [a^{-1}, c]^a[a, c] = [a^{-1}, c][[a^{-1}, c], a][a, c]$, we see that $[a^{-1}, c]$ is of the form $[a, c']$ for some $c' \in C$ and hence $T \triangleleft S$. Since S modulo T is clearly abelian it follows that $S' = T$ as we wished to show. Accordingly, in the case under consideration,

$$S' = M = [a, G'] = \{[a, b_0], b_0 \in B_0\}.$$

Hence since $a^{-1}b \in M$, there is $b_0 \in B_0$ so that $a^{-1}b = [a, b_0]$ and hence $a^{b_0} = b$. Thus a and b are conjugates, and if C_a, C_b denote the centralizers respectively of a and b , then $A \leq C_a, B \leq C_b$ and $G = AB = C_aC_b$. Since C_a and C_b are conjugates, G can equal C_aC_b only if $C_a = G$ (in general if $G = FH$ with $F = H^g, g = hf^{-1}, h \in H, f \in F$, then $F = F' = H^h = H$ and $G = F$). Consequently, a is central in G contrary to the fact that $[a, B_M] \neq 1$.

We conclude that $A_m B_m$ is abelian and hence that M is isomorphic to A_M . Then from its minimality M is finite. By factoring out M we see that modulo M , a minimal normal subgroup of G contained in G' , is again finite. Since the chain is finite, we may conclude that G' is finite. Since G' is finite, each $g \in G$ has a finite number of conjugates and hence the centralizer C_g has finite index in G . Then since G is finitely generated, the center of G has finite index in G and hence intersects A or B nontrivially (G is infinite by assumption). Thus A or B has a central subgroup of G and the proof of (I) is complete.

REMARK. A proof of (I) for G finite can be based on the fact that G has exactly one conjugacy class of Carter (i.e. self-normalizing nilpotent) subgroups. For then not both A and B can be Carter subgroups and hence, without loss of generality, A is properly contained in its normalizer N . Let D denote $N \cap G'$. If $D=1$, then $N \cong NG'/G'$ and is abelian; while if $D>1$, then $[A, D] \leq A \cap G' = 1$ and hence AD is abelian. In either case N contains an abelian subgroup properly containing A and hence an element ab with $a \in A$, $b \in B$, $b \neq 1$ such that $[b, A] = 1$ so that (b) is central in G .

PROOF OF (II). We proceed by induction first on the sum of the torsion-free ranks of A and B and then on the sum of the orders of the torsion subgroups of A and B . In view of (I) there is a normal subgroup N of G contained without loss of generality in A . Then $G/N = A/N \cdot BN/N$ and by the induction assumption (either the torsion-free rank or the order of the torsion subgroup of A/N is less than the corresponding quantity of A), there is a subgroup M of G containing N so that M/N is a proper normal subgroup of G/N containing either A/N or BN/N . It follows that M is a proper normal subgroup of G containing one of the subgroups A or B and (II) is proved.

An induction as in the proof of (II) above easily gives the following.

COROLLARY 1. *If G is the general product AB with A and B finitely generated abelian subgroups, then G is polycyclic; that is, G is solvable and satisfies the maximal condition for subgroups.*

COROLLARY 2. *If G is the general product AB with A and B cyclic, then G is supersolvable.*

BIBLIOGRAPHY

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