SYMMETRIC OPERATORS WITH SINGULAR SPECTRAL FUNCTIONS

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In a previous article [3] it was shown that if A is a closed symmetric operator with deficiency indices $(1, 1)$ in the Hilbert space H and if μ_0 is a point of regular type for A , then there is a neighborhood of μ_0 in which every minimal selfadjoint dilation A^+ of A has spectral multiplicity not more than 1. This shows that in Theorem 14 of Coddington [2] and Theorem 5.2 of McKelvey [6] it is the hypothesis that every point is of regular type which determines the spectral multiplicity of the dilation; the hypothesis that the contraction $F(\lambda)$ is continuous down to the real axis and has norm less than 1 there serves to make the spectral function absolutely continuous. (See Remark 2.)

The procedure used in [3] depended on the fact that if μ_0 is a point of regular type for A , then there exists a selfadjoint extension A_0 of A for which μ_0 is in the resolvent set. This means that for different points μ_0 one might have to choose different operators A_0 . In the present note this necessity is eliminated. It is shown that if A has a selfadjoint extension A_0 with a pure point spectrum with no finite limit points, then for each minimal selfadjoint extension or dilation A^+ of A there is defined on the real axis a nondecreasing function $\rho(\sigma)$ (which depends solely on A^+ and on the single selfadjoint extension A_0) such that A^+ is unitarily equivalent to the multiplication operator in L^2_ρ . Kreĭn [5] has shown that if A is simple, then there exists a selfadjoint extension A_0 of the type described above if and only if every point is of regular type for A . (See Remark 1 also.)

In the case that A is a singular Sturm-Liouville operator, the results of the paper are extended to the case that there exists a selfadjoint extension A_0 with a singular spectral function.

For terminology used we refer the reader to [3] and to Achieser and Glasmann [1].

We note first of all the following lemma due to Štraus [8].

LEMMA 1. *Suppose $\Phi(\lambda)$ is analytic in the upper halfplane with non-negative imaginary part. Suppose $\Psi(\lambda)$ is analytic in some open set containing the real interval $[\alpha, \beta]$. Then,*

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$$(2\pi i)^{-1} \lim_{\tau \rightarrow +0} \int_{\alpha}^{\beta} \{ \Phi(\sigma + i\tau) \Psi(\sigma + i\tau) - [\Phi(\sigma + i\tau)]^{-} \Psi(\sigma - i\tau) \} d\sigma \\ = \int_{\alpha}^{\beta} \Psi(\sigma) d\rho(\sigma), \text{ where } \rho(\sigma) = (1/\pi) \lim_{\tau \rightarrow +0} \int_0^{\sigma} I\Phi(\sigma + i\tau) d\sigma.$$

Here I stands for the imaginary part and $[\]^{-}$ stands for complex conjugate.

We note further the following lemma due to Kreĭn. (See Achieser and Glasmann [1, Appendix I, §4].)

LEMMA 2. *Suppose that A is a closed symmetric operator with deficiency indices $(1, 1)$ in the Hilbert space H . Let A_0 be any selfadjoint extension of A . Let $g(\lambda) = g_0 + (\lambda - \lambda_0)R_0(\lambda)g_0$, where λ_0 is a complex number with positive imaginary part; g_0 is an element of norm 1 in the deficiency subspace of A corresponding to $\bar{\lambda}_0$; $R_0(\lambda)$ is the resolvent of A_0 . Let $Q_1(\lambda) = i\Gamma\lambda_0 + (\lambda - \lambda_0)(g_0, g(\bar{\lambda}))$. Then for $\Gamma\lambda \neq 0$ the generalized resolvent $R(\lambda)$ of A corresponding to a selfadjoint extension or dilation A^+ of A has the form*

$$(1) \quad R(\lambda) = R_0(\lambda) - [\theta(\lambda) + Q_1(\lambda)]^{-1}(\cdot, g(\bar{\lambda}))g(\lambda),$$

where $\theta(\lambda)$ is an analytic function for $\Gamma\lambda \neq 0$ which has nonnegative imaginary part in the upper half-plane. $R(\lambda)$ corresponds to a selfadjoint extension in the space H if and only if $\theta(\lambda)$ is identically equal to a constant θ , where $-\infty < \theta \leq +\infty$. The case $\theta = +\infty$ gives the resolvent $R_0(\lambda)$ corresponding to A_0 .

THEOREM. *Let A be a simple closed symmetric operator with deficiency indices $(1, 1)$ in the Hilbert space H . Let A_0 be a selfadjoint extension of A with a pure point spectrum having no finite limit points. Let A^+ be a minimal selfadjoint extension of A with corresponding generalized resolvent of A represented as in Lemma 2. Then, A^+ is unitarily equivalent to the multiplication operator in L^2_{ρ} , where*

$$(2) \quad \rho(\sigma) = (1/\pi) \lim_{\tau \rightarrow +0} \int_0^{\sigma} I\Phi(\mu + i\tau) d\mu,$$

$$(3) \quad \Phi(\lambda) = [\theta(\lambda)Q_1(\lambda) - 1][\theta(\lambda) + Q_1(\lambda)]^{-1}.$$

If $E^+(\lambda)$ is the spectral function of A^+ , then for all elements $f, h \in H$,

$$(4) \quad ([E^+(\beta) - E^+(\alpha)]f, h) = \int_{[\alpha, \beta)} (f, g(\sigma))(g(\sigma), h)[Q_1^2(\sigma) + 1]^{-1} d\rho(\sigma),$$

where the integral is a Lebesgue-Stieltjes integral.

PROOF. For any f, h in H , Kreĭn's formula (1) for the generalized resolvent $R(\lambda)$ corresponding to A^+ can be written in the form

$$(5) \quad (R(\lambda)f, h) = F_1(\lambda) + F_2(\lambda)[\theta(\lambda)Q_1(\lambda) - 1][\theta(\lambda) + Q_1(\lambda)]^{-1},$$

where

$$(6) \quad F_1(\lambda) = \{(R_0(\lambda)f, h)[Q_1^2(\lambda) + 1] - Q_1(\lambda)(f, g(\bar{\lambda}))(g(\lambda), h)\}[Q_1^2(\lambda) + 1]^{-1}$$

and

$$(7) \quad F_2(\lambda) = (f, g(\bar{\lambda}))(g(\lambda), h)[Q_1^2(\lambda) + 1]^{-1}.$$

We claim that $F_1(\lambda)$ and $F_2(\lambda)$ are entire functions of λ .

In order to establish this claim, we observe that the following equations are true:

$$(8) \quad (R_0(\lambda)f, h) = \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} d(E_0(\mu)f, h);$$

$$(9) \quad (f, g(\bar{\lambda})) = (f, g_0) + (\lambda - \bar{\lambda}_0) \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} d(E_0(\mu)f, g_0);$$

$$(10) \quad (g(\lambda), h) = (g_0, h) + (\lambda - \lambda_0) \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} d(E_0(\mu)g_0, h);$$

$$(11) \quad \begin{aligned} Q_1(\lambda) &= iI\lambda_0 + (\lambda - \lambda_0) \\ &+ (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0) \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} d(E_0(\mu)g_0, g_0), \end{aligned}$$

where $E_0(\mu)$ is the spectral function of A_0 .

If μ_0 is a point in the resolvent set of A_0 , then each of the functions $(R_0(\lambda)f, h)$, $(f, g(\bar{\lambda}))$, $(g(\lambda), h)$, $Q_1(\lambda)$ is an analytic function of λ in a neighborhood of μ_0 . Further, $Q_1(\lambda)$ is real for λ real, so that $Q_1^2(\lambda) + 1$ is not zero in this neighborhood for λ sufficiently close to the real axis. It follows that $F_1(\lambda)$ and $F_2(\lambda)$ are analytic in a neighborhood of μ_0 .

Suppose μ_0 is an eigenvalue of A_0 . Since A is simple, g_0 is a generating element for A_0 . (See Achieser and Glasmann [1, §81, Theorem 2].) Hence we may write equations (8), (9), (10), (11) in the form

$$(12) \quad \begin{aligned} (R_0(\lambda)f, h) &= G_1(\lambda) \\ &+ (\mu_0 - \lambda)^{-1}(f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h)(E_0[\mu_0]g_0, g_0)^{-1}, \end{aligned}$$

$$(13) \quad (f, g(\bar{\lambda})) = G_2(\lambda) + (\lambda - \bar{\lambda}_0)(\mu_0 - \lambda)^{-1}(f, E_0[\mu_0]g_0),$$

$$(14) \quad (g(\lambda), h) = G_3(\lambda) + (\lambda - \lambda_0)(\mu_0 - \lambda)^{-1}(E_0[\mu_0]g_0, h),$$

$$(15) \quad Q_1(\lambda) = G_4(\lambda) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(\mu_0 - \lambda)^{-1}(E_0[\mu_0]g_0, g_0).$$

Here $G_1(\lambda)$, $G_2(\lambda)$, $G_3(\lambda)$ and $G_4(\lambda)$ are analytic in a neighborhood of μ_0 . By the symbol $E_0[\mu_0]$ we denote $E_0(\mu_0 + 0) - E_0(\mu_0 - 0)$.

Using equations (12), (13), (14) and (15), we see that

$$(16) \quad Q_1^2(\lambda) + 1 = G_5(\lambda)(\mu_0 - \lambda)^{-1} + (\lambda - \lambda_0)^2(\lambda - \bar{\lambda}_0)^2(\mu_0 - \lambda)^{-2} \\ \cdot (E_0[\mu_0]g_0, g_0)^2,$$

$$(17) \quad (R_0(\lambda)f, h)[Q_1^2(\lambda) + 1] \\ = G_6(\lambda)(\mu_0 - \lambda)^{-2} + (\lambda - \lambda_0)^2(\lambda - \bar{\lambda}_0)^2 \\ \cdot (\mu_0 - \lambda)^{-3}(f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h)(E_0[\mu_0]g_0, g_0),$$

$$(18) \quad (f, g(\bar{\lambda}))(g(\lambda), h) = G_7(\lambda)(\mu_0 - \lambda)^{-1} + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(\mu_0 - \lambda)^{-2} \\ \cdot (f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h),$$

$$(19) \quad Q_1(\lambda)(f, g(\bar{\lambda}))(g(\lambda), h) \\ = G_8(\lambda)(\mu_0 - \lambda)^{-2} + (\lambda - \lambda_0)^2(\lambda - \bar{\lambda}_0)^2 \\ \cdot (\mu_0 - \lambda)^{-3}(f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h)(E_0[\mu_0]g_0, g_0),$$

where $G_5(\lambda)$, $G_6(\lambda)$, $G_7(\lambda)$, $G_8(\lambda)$ are analytic in a neighborhood of μ_0 .

Substituting (16), (17), (18), (19) into (6) and (7), we obtain

$$F_1(\lambda) = [G_6(\lambda) - G_8(\lambda)] \\ \cdot [G_5(\lambda)(\mu_0 - \lambda) + (\lambda - \lambda_0)^2(\lambda - \bar{\lambda}_0)^2(E_0[\mu_0]g_0, g_0)^2]^{-1}, \\ F_2(\lambda) = [G_7(\lambda)(\mu_0 - \lambda) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h)] \\ \cdot [G_5(\lambda)(\mu_0 - \lambda) + (\lambda - \lambda_0)^2(\lambda - \bar{\lambda}_0)^2(E_0[\mu_0]g_0, g_0)^2]^{-1}.$$

From these expressions we see that $F_1(\lambda)$ and $F_2(\lambda)$ are analytic in a neighborhood of an eigenvalue μ_0 .

This establishes our claim that $F_1(\lambda)$ and $F_2(\lambda)$ are entire functions of λ .

Let $E(\mu)$ be the spectral function of A corresponding to the self-adjoint extension or dilation A^+ . By the Stieltjes inversion formula, for all f, h in H and for any real numbers α, β , $\alpha < \beta$,

$$(20) \quad \left(\left\{ \frac{1}{2} \{ E(\beta) + E(\beta + 0) \} - \frac{1}{2} \{ E(\alpha) + E(\alpha + 0) \} \right\} f, h \right) \\ = (2\pi i)^{-1} \lim_{\tau \rightarrow +0} \int_{\alpha}^{\beta} ([R(\sigma + i\tau) - R(\sigma - i\tau)]f, h) d\sigma.$$

Substituting (5) into (20), we obtain

$$\begin{aligned}
& ([\tfrac{1}{2}\{E(\beta) + E(\beta + 0)\} - \tfrac{1}{2}\{E(\alpha) + E(\alpha + 0)\}]f, h) \\
&= (2\pi i)^{-1} \lim_{\tau \rightarrow +0} \int_{\alpha}^{\beta} [F_1(\sigma + i\tau) - F_1(\sigma - i\tau)] d\sigma \\
&+ (2\pi i)^{-1} \lim_{\tau \rightarrow +0} \int_{\alpha}^{\beta} \{F_2(\sigma + i\tau)[\theta(\sigma + i\tau)Q_1(\sigma + i\tau) - 1] \\
&\quad \cdot [\theta(\sigma + i\tau) + Q_1(\sigma + i\tau)]^{-1} \\
&\quad - F_2(\sigma - i\tau)[\theta(\sigma - i\tau)Q_1(\sigma - i\tau) - 1] \\
&\quad \cdot [\theta(\sigma - i\tau) + Q_1(\sigma - i\tau)]^{-1}\} d\sigma.
\end{aligned}$$

The first limit is zero by the analyticity of $F_1(\lambda)$. Since $\theta(\bar{\lambda}) = [\theta(\lambda)]^-$, $Q_1(\bar{\lambda}) = [Q_1(\lambda)]^-$ and $IQ_1(\lambda) > 0$ for $I\lambda > 0$ (see [3]), we may put $\Psi(\lambda) = F_2(\lambda)$ and $\Phi(\lambda) = [\theta(\lambda)Q_1(\lambda) - 1][\theta(\lambda) + Q_1(\lambda)]^{-1}$ in Lemma 1 in order to evaluate the second limit. Hence,

$$\begin{aligned}
& ([\tfrac{1}{2}\{E^+(\beta) + E^+(\beta + 0)\} - \tfrac{1}{2}\{E^+(\alpha) + E^+(\alpha + 0)\}]f, h) \\
&= ([\tfrac{1}{2}\{E(\beta) + E(\beta + 0)\} - \tfrac{1}{2}\{E(\alpha) + E(\alpha + 0)\}]f, h) \\
(21) \quad &= \int_{\alpha}^{\beta} (f, g(\sigma))(g(\sigma), h)[Q_1^2(\sigma) + 1]^{-1} d\rho(\sigma),
\end{aligned}$$

where $\rho(\sigma) = (1/\pi) \lim_{\tau \rightarrow +0} \int_0^{\sigma} I\Phi(\mu + i\tau) d\mu$, and $E^+(\mu)$ is the spectral function of A^+ .

It is understood that the integrand is defined by continuity at the eigenvalues of A_0 . From (21) follows equation (4).

Since A^+ is a minimal selfadjoint extension, the Hilbert space H^+ in which A^+ acts is the closed linear hull of the set Z of elements of the form $[E^+(\beta) - E^+(\alpha)]f$ for arbitrary α, β and arbitrary f in H . (See Naimark [7].)

Let $\{\mu_k\}$ be the eigenvalues of A_0 arranged in order of growth. If $|k|$ is even, let $Q_2(\sigma) = [Q_1^2(\sigma) + 1]^{1/2}$ for $\mu_k < \sigma < \mu_{k+1}$; if $|k|$ is odd, let $Q_2(\sigma) = -[Q_1^2(\sigma) + 1]^{1/2}$ for $\mu_k < \sigma < \mu_{k+1}$. If the eigenvalues are bounded above or below, we modify the above definition accordingly for σ greater than the largest eigenvalue or for σ less than the smallest eigenvalue. The definition of $Q_2(\sigma)$ has been contrived so that $(f, g(\sigma))[Q_2(\sigma)]^{-1}$ is a continuous function of σ .

We define an operator V on Z into L_{ρ}^2 as follows: $V[E^+(\beta) - E^+(\alpha)]f = \chi_{[\alpha, \beta]}(\sigma)(f, g(\sigma))[Q_2(\sigma)]^{-1}$, where $\chi_{[\alpha, \beta]}(\sigma)$ is the characteristic function of $[\alpha, \beta]$. From (4) we see that $\|V[E^+(\beta) - E^+(\alpha)]f\| = \|[E^+(\beta) - E^+(\alpha)]f\|$. We now extend V linearly to the linear hull of Z and by continuity to all of H^+ . V is then an isometry of H^+ into L_{ρ}^2 . V is in fact onto, because the set VZ is dense in L_{ρ}^2 . To see this, sup-

pose that $k(\sigma) \in L^2_\rho$ and that $k(\sigma)$ is perpendicular to $\chi_{[\alpha, \beta]}(\sigma)(f, g(\sigma)) \cdot [Q_2(\sigma)]^{-1}$ for all intervals $[\alpha, \beta]$ and for all f in H . As in [3] it follows that $k(\sigma) = 0$ almost everywhere with respect to ρ for points in the resolvent set of A_0 . If μ_0 is an eigenvalue of A_0 , we can show that

$$\int_{\{\mu_0\}} (f, g(\sigma)) [Q_2(\sigma)]^{-1} k(\sigma) d\rho(\sigma) = 0,$$

i.e.,

$$(f, g(\mu_0)) [Q_2(\mu_0)]^{-1} k(\mu_0) \Delta\rho(\mu_0) = 0,$$

where $\Delta\rho(\mu_0)$ is the ρ -measure of μ_0 . Taking $f = g_0$ in this last equation and using the fact that $(g_0, g(\mu_0)) [Q_2(\mu_0)]^{-1} \neq 0$, we see that $k(\mu_0) = 0$ if $\Delta\rho(\mu_0) \neq 0$. (Note that by $(f, g(\mu_0)) [Q_2(\mu_0)]^{-1}$ we mean the value of the function $(f, g(\sigma)) [Q_2(\sigma)]^{-1}$ defined by continuity at $\sigma = \mu_0$.) Hence, $k(\sigma) = 0$ almost everywhere with respect to ρ . This shows that VZ is dense in L^2_ρ .

It is not difficult to check that V carries the spectral function of A^+ into the spectral function of the multiplication operator in L^2_ρ . This proves the theorem.

REMARK 1. Suppose A^+ is a selfadjoint extension of A in H (different from A_0), so that the function $\theta(\lambda)$ in equation (3) is identically equal to a real constant θ_0 . Then from equation (2) for $\rho(\sigma)$ and the fact that A^+ is unitarily equivalent to the multiplication operator in L^2_ρ , we easily deduce the well-known results that the spectrum of A^+ consists of eigenvalues with no finite limit point and that the eigenvalues of A^+ are all different from those of A_0 . From this it follows that every point is of regular type for A .

REMARK 2. Let $F(\lambda) = [\theta(\lambda) - iI\lambda_0][\theta(\lambda) + iI\lambda_0]^{-1}$ for $I\lambda > 0$. Then $F(\lambda)$ is analytic in the upper half-plane and $|F(\lambda)| \leq 1$. Suppose that $F(\lambda)$ is continuous down to the real axis and that $|F(\lambda)| < 1$ on the real axis. Then one may check that $\Phi(\lambda)$ is continuous (i.e., may be defined continuously) down to the real axis, and $I\Phi(\lambda)$ is positive on the real axis. Hence, $\rho(\mu) = (1/\pi) \int_0^\infty I\Phi(\mu) d\mu$ is absolutely continuous, and the multiplication operator in L^2_ρ is unitarily equivalent to the multiplication operator in L^2 . Thus, if $F(\lambda)$ is continuous down to the real axis and $|F(\lambda)| < 1$ on the real axis, it follows that A^+ is unitarily equivalent to the multiplication operator in L^2 .

REMARK 3. If A is a singular Sturm-Liouville operator, one may use the work of Kac [4, Theorem 7] in order to consider the case that there exists a selfadjoint extension A_0 with a singular spectral function. Let A be the symmetric operator in $L^2[0, \infty)$ generated by the differential operator $l[y] = -d^2y/dx^2 + q(x)y$ where q is real and con-

tinuous for $0 \leq x < \infty$. Suppose that the differential operator is in the limit point case and that the spectral function $\rho_0(\sigma)$ corresponding to the Weyl function $m(\lambda)$ is singular (i.e., A has a selfadjoint extension A_0 with a singular spectral function). According to Štraus [9] there is a one-one correspondence between the minimal selfadjoint dilations A^+ of A and the class of functions $\theta(\lambda)$ which are analytic in the upper half-plane and have nonnegative imaginary part. If $E(\lambda)$ is the spectral function of A corresponding to the minimal selfadjoint extension A^+ , then by the work of Kac, for $f \in L^2(0, \infty)$,

$$\begin{aligned} & \left[\frac{1}{2} \{ E(\beta) + E(\beta + 0) \} - \frac{1}{2} \{ E(\alpha) + E(\alpha + 0) \} \right] f(x) \\ &= \int_{\alpha}^{\beta} \int_0^{\infty} f(y) u(y, \sigma) dy u(x, \sigma) d\rho(\sigma) \end{aligned}$$

where

$$\rho(\sigma) = (1/\pi) \lim_{\tau \rightarrow +0} \int_0^{\sigma} I\Phi(\mu + i\tau) d\mu,$$

$$\Phi(\lambda) = [\theta(\lambda)m(\lambda) - 1] [\theta(\lambda) + m(\lambda)]^{-1},$$

and $u(x, \sigma)$ is a solution of $l[y] - \sigma y = 0$, $y(0) + m(\sigma)y'(0) = 0$. The latter equation is to be interpreted as $y'(0) = 0$ in the event that $\lim_{\lambda \rightarrow 0} m(\lambda)$ does not exist as a finite number.

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