ON A CLASS OF FINER TOPOLOGIES WITH THE SAME CLASS OF HOMEOMORPHISMS¹

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Given a topological space $(X, \mathfrak{U})$, let $H(X, \mathfrak{U})$ be the class of all homeomorphisms of $(X, \mathfrak{U})$ onto itself. Everett and Ulam [1], [5] posed the following problem. What topologies $\mathfrak{V}$ exist on X such that $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$, i.e., these two topological spaces have exactly the same class of mappings as homeomorphisms.

Many topologies $\mathfrak{V}$ on X such that $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$ have been constructed in [3]. However, all topologies constructed in [3] are coarser than the original topology. In general, we cannot reverse the procedure to reconstruct the topologies $\mathfrak{U}$ out of $\mathfrak{V}$ and to prove that they possess the same class of mappings as homeomorphisms, because the topologies $\mathfrak{V}$ constructed in [3] are hard to characterize. Therefore it is natural to ask the question, given a topological space $(X, \mathfrak{U})$, can we construct topologies $\mathfrak{V} \supsetneq \mathfrak{U}$ such that $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$. This paper is devoted to investigating this problem.

By $X \setminus A$, $\text{Cl}(A)$ and $\text{Int}(A)$, unless otherwise stated, we always mean the complement, closure and the interior of A relative to the original topology $\mathfrak{U}$. We denote the neighborhood system of a point p with respect to $\mathfrak{U}$ by $\mathfrak{U}_p$.

A family of subsets $\mathfrak{X}$ in $(X, \mathfrak{U})$ is called an $\mathfrak{F}$ -family if the following three conditions are satisfied:

- (1) The empty set $\emptyset$ is in $\mathfrak{X}$.
- (2) If $N \in \mathfrak{X}$ then $\text{Int}(N) = \emptyset$.
- (3) If $\{N_1, \dots, N_k\} \subset \mathfrak{X}$ and $\{f_1, \dots, f_k\} \subset H(X, \mathfrak{U})$, then $\bigcup \{f_i(N_i) : i = 1, \dots, k\} \in \mathfrak{X}$ for every k .

An $\mathfrak{F}$ -family $\mathfrak{X}$ is said to be a strong $\mathfrak{F}$ -family if $\text{Int}(\text{Cl}(N)) = \emptyset$ for all N in $\mathfrak{X}$. We shall use the symbol $\text{Cl}(A, \mathfrak{V})$ to denote the closure of A with respect to a new topology $\mathfrak{V}$.

The following lemma may be derived straightforwardly from the definition of an $\mathfrak{F}$ -family:

LEMMA 1. *If $\mathfrak{X}$ is an $\mathfrak{F}$ -family of $(X, \mathfrak{U})$ then $\{U - N : U \in \mathfrak{U}, N \in \mathfrak{X}\}$ forms a base of a topology $\mathfrak{V}$ on X . In this case, we say that $\mathfrak{V}$ is generated by $\mathfrak{X}$.*

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THEOREM 1. *Let $\mathfrak{X}$ be an $\mathfrak{S}$ -family of subsets of a regular T_1 -space $(X, \mathfrak{U})$. Let $\mathfrak{V}$ be the topology generated by $\mathfrak{X}$. Then $\mathfrak{U} \subset \mathfrak{V}$ and $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$.*

PROOF. It is clear that $H(X, \mathfrak{U}) \subset H(X, \mathfrak{V})$. Suppose there exists f in $H(X, \mathfrak{V}) - H(X, \mathfrak{U})$. Then either there exists a $V \notin \mathfrak{U}_{p_0}$ with $f(V) \in \mathfrak{U}_{f(p_0)}$, or $V \notin \mathfrak{U}_{p_0}$ but $f^{-1}(V) \in \mathfrak{U}_{f^{-1}(p_0)}$ for some p_0 . We need only show that the first alternative leads to a contradiction.

Clearly p_0 is not an isolated point. Since $f \in H(X, \mathfrak{V})$ and $f(V) \in \mathfrak{U}_{f(p_0)} \subset \mathfrak{V}_{f(p_0)}$, we have $V \in \mathfrak{V}_{p_0}$. Hence there exists $U \in \mathfrak{U}$ and $N \in \mathfrak{X}$ such that

$$p_0 \in U - N \subset V \quad \text{and} \quad p_0 \in \text{Cl}(N) - N.$$

Since $f(p_0) \in f(V) \in \mathfrak{U}_{f(p_0)}$ and $(X, \mathfrak{U})$ is regular, there exists $f(V_1) \in \mathfrak{U}_{f(p_0)}$ such that $\text{Cl}(f(V_1)) \subset f(V)$. Since $f(V_1) \in \mathfrak{U}_{f(p_0)} \subset \mathfrak{V}_{f(p_0)}$, it follows that $V_1 \in \mathfrak{V}_{p_0}$ and $V_1 \supset U_1 - N_1$ for some $U_1 \in \mathfrak{U}_{p_0} \cap \mathfrak{U}$, $N_1 \in \mathfrak{X}$ and $p_0 \in \text{Cl}(N_1) - N_1$. Thus $[(N \cap U) - V] \cap [(N_1 \cap U_1) - V] \neq \emptyset$, since otherwise we would have $V \supset U \cap U_1 \in \mathfrak{U}_{p_0}$. Let $q \in [(N \cap U) - V] \cap [(N_1 \cap U_1) - V]$. Then $f(q) \in f[(N \cap U) - V] \subset X - f(V) \subset X - \text{Cl}(f(V_1))$, and thus $f(q) \notin \text{Cl}(f(V_1), \mathfrak{V})$.

But $q \in [(N_1 \cap U_1) - V]$; hence if $q \in \text{Cl}(U_1 - N_1, \mathfrak{V})$, then there exists $U_q - N_2 \in \mathfrak{V}_q$ such that $U_q \in \mathfrak{U}_q \cap \mathfrak{U}$, $N_2 \in \mathfrak{X}$ and

$$(U_q - N_2) \cap (U_1 - N_1) = (U_q \cap U_1) - (N_2 \cup N_1) = \emptyset.$$

Since U_1 is open, $\emptyset \neq U_q \cap U_1 \in \mathfrak{U} \cap \mathfrak{U}_q$ and $U_q \cap U_1 \subset N_2 \cup N_1$, which would imply that $N_2 \cup N_1$ is somewhere dense. Therefore

$$q \in \text{Cl}(U_1 - N_1, \mathfrak{V}) \subset \text{Cl}(V_1, \mathfrak{V}).$$

Hence $f(q) \in \text{Cl}(f(V_1), \mathfrak{V})$. This contradiction shows that $f(V) \in \mathfrak{U}_{f(p_0)}$ implies $V \in \mathfrak{U}_{p_0}$ for each p_0 in X . Thus f is continuous. Similarly f^{-1} is continuous and we have $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$.

If $\mathfrak{X}$ contains an element which is not closed, then $\mathfrak{U} \subsetneq \mathfrak{V}$. If further, $(X, \mathfrak{U})$ is locally Euclidean, then $(X, \mathfrak{U})$ is not homeomorphic to $(X, \mathfrak{V})$ [4].

If $(X, \mathfrak{U})$ is a first countable regular T_1 Baire space and every element in $\mathfrak{X}$ is of the first category, and if $\mathfrak{X}$ contains at least one element which is not closed in $(X, \mathfrak{U})$, then $\mathfrak{U} \subsetneq \mathfrak{V}$ and $(X, \mathfrak{U})$ and $(X, \mathfrak{V})$ are not homeomorphic. To see this, let $N \in \mathfrak{X}$ be a nonclosed set in $(X, \mathfrak{U})$. Then there exists a point p_0 in $X \setminus N$ and a sequence of points $\{p_i: i = 1, 2, \dots\}$ in N which converges to p_0 ; hence $\{p_i: i = 1, 2, \dots\}$ is a closed set in $(X, \mathfrak{V})$. Suppose $(X, \mathfrak{V})$ is regular, then there exist $\{U_i: i = 0, 1, 2, \dots\}$ in $\mathfrak{U}$ and $\{N_i: i = 0, 1, 2, \dots\}$

in $\mathfrak{X}$ such that $p_i \in U_i \setminus N_i$ for $i=0, 1, 2, \dots$ and $\emptyset = (U_0 \setminus N_0) \cap \bigcup_{i=1}^{\infty} (U_i \setminus N_i) = \bigcup_{i=1}^{\infty} (U_0 \cap U_i \setminus N_0 \cup N_i)$.

Therefore $U_0 \cap U_i \subset N_0 \cup N_i$ for each $i=1, 2, \dots$. But U_0 contains all but finitely many p_i . Therefore there exists $i \neq 0$ such that $U_0 \cap U_i \neq \emptyset$. Hence $N_0 \cup N_i$ contains nonvoid interior which is contradictory to the assumption that every element in $\mathfrak{X}$ is of the first category and $(X, \mathfrak{U})$ is a Baire space. Hence $(X, \mathfrak{V})$ is not regular and $(X, \mathfrak{U})$ and $(X, \mathfrak{V})$ are not homeomorphic.

LEMMA 2. *The intersection of a collection of $\mathfrak{F}$ -families of $(X, \mathfrak{U})$ is an $\mathfrak{F}$ -family of $(X, \mathfrak{U})$.*

The proof is straightforward.

LEMMA 3. *If $\{\mathfrak{X}_\alpha: \alpha \in B\}$ is a collection of $\mathfrak{F}$ -families with at most one not strong $\mathfrak{F}$ -family, then there exists a smallest $\mathfrak{F}$ -family containing $\bigcup \{\mathfrak{X}_\alpha: \alpha \in B\}$, namely the family of all finite unions of members of $\bigcup \{\mathfrak{X}_\alpha: \alpha \in B\}$.*

PROOF. The conditions (1) and (3) are clearly satisfied. Let $\mathfrak{X}_1$ be the only possible nonstrong $\mathfrak{F}$ -family. Let $N_1 \in \mathfrak{X}_1$ and $N_2 \in \mathfrak{X}_\alpha$. Suppose $\text{Int}(N_1 \cup N_2) = U \neq \emptyset$. Then $\text{Int}(N_1 \cup \text{Cl}(N_2)) \supset U$. Since $\text{Int Cl}(N_2) = \emptyset$, $\emptyset \neq U \setminus \text{Cl}(N_2)$ is open and

$$U \setminus \text{Cl}(N_2) \subseteq (N_1 \cup \text{Cl}(N_2)) \setminus \text{Cl}(N_2) \subseteq N_1.$$

This is a contradiction, since $\text{Int}(N_1) = \emptyset$. Similarly we can prove this for any finite union of members of $\bigcup \{\mathfrak{X}_i: i=1, 2, \dots, m\}$.

The condition that all but at most one of the collection of $\mathfrak{F}$ -families $\{\mathfrak{X}_\alpha: \alpha \in B\}$ are strong is necessary as illustrated in the following example.

Let $(X, \mathfrak{U})$ be the rigid continuum of J. J. De Iongh [2, p. 443], then $H(X, \mathfrak{U})$ consists of one element, namely the identity mapping from X onto X . Let x_0 be a fixed point in X which is different from the end point. Let $A = \{x \in X: d(x, x_0) \text{ is a rational number}\}$ and $B = X \setminus A$. Then $X = A \cup B$, $A \cap B = \emptyset$, $\text{Int}(A) = \text{Int}(B) = \emptyset$ and $\text{Cl}(A) = \text{Cl}(B) = X$. Then the family $P(A)$ of all subsets of A and the family $P(B)$ of all subsets of B are both nonstrong $\mathfrak{F}$ -families, and there does not exist an $\mathfrak{F}$ -family which contains $P(A)$ and $P(B)$.

LEMMA 4. *If $\{\mathfrak{X}_\alpha: \alpha \in D\}$ is a collection of $\mathfrak{F}$ -families of $(X, \mathfrak{U})$, and $\{A_\alpha: \alpha \in D\}$ is a family of mutually disjoint closed sets such that $N_\alpha \subset A_\alpha$ for each $N_\alpha \in \mathfrak{X}_\alpha$, then there exists a smallest $\mathfrak{F}$ -family containing $\bigcup \{\mathfrak{X}_\alpha: \alpha \in D\}$, namely the family of all finite unions of members of $\bigcup \{\mathfrak{X}_\alpha: \alpha \in D\}$.*

PROOF. Let $\mathfrak{X}$ be the class of all finite unions of elements of $\{N: N \in \mathfrak{X}_\alpha, \alpha \in D\}$. Then the conditions (1) and (3) are clearly satisfied. Let $\{N_1 \in \mathfrak{X}_\alpha, N_2 \in \mathfrak{X}_\beta\}$ for some α and β in D and suppose $U = \text{Int}(N_1 \cup N_2)$. If $p \in U \cap N_1$, since $\text{Int}(N_1) = \emptyset$, we have $p \in \text{Cl}(N_2)$ and $U \cap N_1 \subset \text{Cl}(N_2)$. Also $U \cap N_1 \subset N_1$. We have $U \cap N_1 \subset \text{Cl}(N_1) \cap \text{Cl}(N_2)$. Similarly, $U \cap N_2 \subset \text{Cl}(N_1) \cap \text{Cl}(N_2)$. Therefore $U \subset \text{Cl}(N_1) \cap \text{Cl}(N_2) \subset A_1 \cap A_2 = \emptyset$. By induction we have (2).

From the above lemmas and Theorem 1, we have the following theorem which follows immediately.

THEOREM 2. Let $\{\mathfrak{X}_\alpha: \alpha \in D\}$ be a class of $\mathfrak{F}$ -families of subsets of a regular T_1 -space $(X, \mathfrak{U})$ and let $\mathfrak{V}$ be the topology generated by taking the family of subsets $\{U \setminus N: N \in \mathfrak{X}_\alpha, \alpha \in D\}$ as a subbase. If one of the following conditions is satisfied:

(a) the collection $\{\mathfrak{X}_\alpha: \alpha \in D\}$ is a collection of strong $\mathfrak{F}$ -families except possibly one;

(b) there exists a family of mutually disjoint closed sets $\{A_\alpha: \alpha \in D\}$ such that $N_\alpha \subset A_\alpha$ for each $N_\alpha \in \mathfrak{X}_\alpha$; then $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$.

In the following corollary, when we say a sequence of subsets $\{A_i\}$ converges to A , we mean that every neighborhood of A contains all but finitely many A_i . And let $H(A) = \{f(x): f \in H(X, \mathfrak{U}), x \in A\}$. The following corollary serves as an example to illustrate the preceding theorems and concepts.

COROLLARY. Let A be a subset of a regular T_1 -space $(X, \mathfrak{U})$. Let $\mathfrak{X}_1$ be the family of all finite unions of subsets N of the form $\bigcup \{C_i: i=1, 2, 3, \dots\}$, where C_i is a closed (and compact and/or perfect) nowhere dense subset of X and $\{C_i\}$ converges to some p in $H(A)$. Let $\mathfrak{X}_2$ be the family of all finite unions of subsets N of the form $\{p_i: i=1, 2, 3, \dots\}$, where $\{P_i\}$ converges to some p in $H(A)$ and p_i is not an isolated point for each i . Let $\mathfrak{X}_3$ be the family of all finite unions of subsets N of the form $\bigcup \{C_i: i=1, 2, \dots\}$, where C_i is a closed connected (and compact) nowhere dense set and C_i separates X , and C_i converges to some p in $H(A)$. If $\mathfrak{V}_i$ is the topology generated by taking the family of all subsets of the form $U - N$, where $U \in \mathfrak{U}$ and $N \in \mathfrak{X}_i$ as a base ($i=1, 2, 3$), then $H(X, \mathfrak{U}) = H(X, \mathfrak{V}_i)$ ($i=1, 2, 3$).

PROOF. To prove the corollary, we only need to verify that $\mathfrak{X}_i$ ($i=1, 2, 3$) is an $\mathfrak{F}$ -family. We do it for $n=1$, and the other cases are similar.

Clearly $\emptyset \in \mathfrak{X}_1$. Let $\{C_i\}$ converge to some p in $H(A)$ with each C_i a closed nowhere dense subset of X . Let U be the interior of $\bigcup \{C_i: i=1, 2, \dots\}$. If $U \neq \emptyset$, then $U \setminus \{p\} \neq \emptyset$ since otherwise p

would be an interior point of C_i for some i . Choose q in $U \setminus \{p\}$. Since X is a regular T_1 -space, there exist disjoint open sets V and W such that $q \in V \subset U \setminus \{p\}$ and $p \in W$. Therefore W contains all but finitely many C_i . Since each C_i is nowhere dense the interior of a subset of any finite union of C_i is void. But V is contained in a finite union of C_i ; this is a contradiction and we have $U = \emptyset$. It follows that $\mathfrak{X}_1$ satisfies (2). If $\{N_1, \dots, N_k\} \subset \mathfrak{X}_1$ and $\{f_1, \dots, f_k\} \subset H(X, \mathfrak{U})$, without loss of generality, we may assume that for each i , $N_i = \bigcup \{C_{ij} : j=1, 2, \dots\}$ where C_{ij} is a closed nowhere dense subset of X and $\{C_{ij} : j=1, 2, \dots\}$ converges to some p_i in $H(A)$. Then it is clear $\{f_i(C_{ij}) : j=1, 2, \dots\}$ converges to $f_i(p_i)$ which is also in $H(A)$ and $f_i(C_{ij})$ is a closed nowhere dense subset of X for each $i=1, 2, \dots, k$ and $j=1, 2, \dots$. Hence by definition $\bigcup \{f_i(N_i) : i=1, 2, \dots, k\} \in \mathfrak{X}_1$ and $\mathfrak{X}_1$ is a $\mathfrak{S}$ -family. By Theorem 1, we have $H(X, \mathfrak{U}) = H(X, \mathfrak{U}_1)$.

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